On the Nonexistence of Simplex Integration Rules for Infinite Integrals

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Abstract. It is shown that there do not exist integration rules of the form
\[ \int_0^\infty f(x) \, dx = \sum_{i=1}^n w_i f(x_i) + C \left( f^{(m)}(\xi) \right), \quad 0 < \xi < \infty. \]

Almost all classical integration rules over a finite interval are simplex, that is, they have the form
\[ \int_a^b f(x) \, dx = \sum_{i=1}^n w_i f(x_i) + C f^{(k)}(\xi), \quad a < \xi < b, \xi = \xi(f), \]
where \( C \) is a constant depending on the rule and interval, but independent of \( f \), and \( k \) is some integer which is characteristic for the rule. Some special rules, for example Weddle's rule, which are not simplex are multiplex, that is, the error has the form \( \sum_{i=1}^n C f^{(k)}(\xi_i) \). It is the aim of this note to show that there can exist no simplex or multiplex rule for the infinite integral \( \int_0^\infty f(x) \, dx \). Although the Gauss-Laguerre rule
\[ \int_0^\infty e^{-x} f(x) \, dx = \sum_{i=1}^n w_i f(x_i) + C \left( e^{-x} f^{(2n)}(\xi) \right) \]
appears to have the form of a simplex rule, this is not so, since we are concerned with unweighted integrals and if we write \( f(x) = e^{-x} e^x f(x) \), we have that
\[ \int_0^\infty f(x) \, dx = \sum_{i=1}^n w_i e^{x_i} f(x_i) + C \left( e^{x_i} f^{(2n)}(\xi) \right) \]
which is neither simplex nor multiplex in form.

We now show that it is impossible to have an integration rule of the form
\[ Jf \equiv \int_0^\infty f(x) \, dx = \sum_{i=1}^n w_i f(x_i) + C f^{(k)}(\xi), \quad 0 < \xi < \infty, \]
valid for all \( f \in L(0, \infty) \cap C^k(0, \infty) \), or for that matter, one where there are a finite number of terms of the form \( C f^{(k)}(\xi_i) \). The proof is based on the simple fact that, for any \( r > 0 \),
\[ \int_0^\infty f(x) \, dx = r \int_0^\infty f(rx) \, dx. \]

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If (1) were true, then (2) would imply that

\[ If = r \int_{0}^{\infty} f(rx) \, dx = r \sum_{i=1}^{n} w_{i} f(rx_{i}) + rCf^{(k)}(r\xi) \]

(3)

\[ = r \sum_{i=1}^{n} w_{i} f(rx_{i}) + r^{k+1}Cf^{(k)}(\xi), \quad 0 < \xi < \infty, \]

which must hold for all \( f \in L[0, \infty) \cap C^{k}(0, \infty) \) and any real \( r \). If we now choose such a function which is bounded together with its \( k \)th derivative on \([0, \infty)\), say \( f(x) = 1/(1 + x^{2}) \), and let \( r \) approach zero, we see that the right-hand side of (3) approaches zero while the left-hand side has a constant value. This contradiction proves our result.

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