The Asymptotic Expansion of the Meijer $G$-Function*

By Jerry L. Fields

Abstract. Gamma function identities are integrated to expand the Meijer $G$-function in a basic set of functions, each of which is simply characterized asymptotically.

1. Introduction and Notation. In this exposition, we derive the asymptotic expansion of the Meijer $G$-function for large values of the variable. Although these results can be found in various places in the literature, e.g., Meijer’s original papers [1], or their collection by Luke [2], they are usually obscured by a maze of special notation and the presence of a large number of results which are only of secondary interest. The following derivation seems more direct.

Throughout this work, we assume that the integers $p, q, m, n$ and parameters $a_i, b_i$ satisfy the hypothesis,

$$0 \leq m \leq q, \quad 0 \leq n \leq p,$$

(1.1) $a_i - b_k \neq 0$ a positive integer; $j = 1, \ldots, p; k = 1, \ldots, q,$

$$a_i - a_k \neq 0$ an integer; $j, k = 1, \ldots, p; j \neq k.$

Extensive use will be made of the notations,

$$\Gamma_a(c_p - t) = \prod_{k=1}^{p} \Gamma(c_k - t), \quad \Gamma(c_m - t) = \Gamma_0(c_m - t),$$

(1.2) $s F_s\left(\frac{a_p}{b_q} \left| w \frac{1}{t} w^k \right) = \sum_{k=0}^{\infty} \frac{\Gamma(a_p + k)\Gamma(b_q)}{\Gamma(b_q + k)\Gamma(a_p + k)} \frac{w^k}{k!}.$

The Meijer $G$-function is then defined by

$$G_{p,q}^{m,n}(z) = G_{p,q}^{m,n}(z \left| \begin{matrix} a_p \\ b_q \end{matrix} \right| \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix})$$

(1.3) $= \frac{1}{2\pi i} \int_{L} \frac{\Gamma(b_m - t)\Gamma(1 - a_N + t)z^t}{\Gamma(m(1 - b_q + t)\Gamma(a_p - t))} dt,$

where $L$ is an upward oriented loop contour which separates the poles of $\Gamma(b_m - t)$ from those of $\Gamma(1 - a_N + t)$ and which begins and ends at $+\infty$ ($L = L_+$) or $-\infty$.

Received November 20, 1971.

AMS 1970 subject classifications. Primary 33A35; Secondary 33A02.

Key words and phrases. Meijer $G$-functions, asymptotic expansions, linear differential equations, irregular singular points.

* This work was initiated in 1968–69 while the author was a visiting professor at the National Tsing Hua University, Taiwan, Republic of China, and completed under grant NRC A 7549 of the National Research Council of Canada.

Copyright © 1972, American Mathematical Society
(L = L.). A simple computation shows that $G_{p,q}^{n,m}(z)$ satisfies the linear differential equation

\[
\left\{ \prod_{j=1}^{n} (\delta - b_j) + (-1)^{p+1-m-n} z \prod_{j=1}^{p} (\delta + 1 - a_j) \right\} y(z) = 0, \quad \delta = z \frac{d}{dz},
\]

of order $\max(p, q)$.

If $q \leq p$, $z = \infty$ is a regular singular point of (1.4). The behaviour of $G_{p,q}^{n,m}(z)$ for $z$ large then follows from the residue calculus result, $L = L_-$.

**Theorem 1.** Under the conditions of (1.1),

\[
G_{p,q}^{n,m}(z) = \sum_{j=1}^{n} \frac{\Gamma^*(a_i - a_n) \Gamma(1 + b_q - a_i)}{\Gamma_n(1 + a_p - a_i) \Gamma_m(a_i - b_q)} z^{-1+a_j}.
\]

(1.5)

If $n = 0$, (1.5) reduces to $G_{p,q}^{n,0}(z) = 0$.

If $q > p$, $z = \infty$ is an irregular singular point of (1.4), and the analysis is more involved. For convenience, we set

\[
v = q - p \geq 1, \quad \mu = q - m - n.
\]

A special case of the above, $m = q$ and $n = 0$ or 1, was treated by Barnes [3], who obtained

**Theorem 2 (Barnes).** Under the conditions of (1.1), $\nu \geq 1$,

\[
L_j(w) \equiv G_{p,q}^{q,0} \left( w \left| \begin{array}{c} a_1, a_1, \ldots, a_j, a_{j+1}, \ldots, a_p \end{array} \right. \right) \frac{\Gamma(1 + b_q - a_j) \Gamma(1 + a_p - a_j)}{w^{-1+a_j} \Gamma(1 + b_q - a_j) \Gamma(1 + a_p - a_j)} F_{\nu+1} \left( \begin{array}{c} 1, 1 + b_q - a_j \end{array} \right| \frac{w}{1 + a_p - a_j} \left| -1 \right),
\]

(1.8)

$w \to \infty$, \quad |\arg w| < \pi(v/2 + 1),

and

\[
G(w) \equiv G_{p,q}^{q,0} \left( w \left| \begin{array}{c} a_1, \ldots, a_p \end{array} \right. \right) \frac{(2\pi)^{1/2} \nu^{-1/2}}{\nu} \exp(-\nu^{1/2} w^{-1}) \sum_{j=0}^{\infty} K_j w^{\nu - j},
\]

(1.9)

$w \to \infty$, \quad |\arg w| < \pi(v + \min(1, \nu/2)),
asymptotic expansion of the Meijer G-function

\[ \nu \gamma = \frac{1 - \nu}{2} + B_1 - A_1, \quad K_0 = 1, \]

(1.10)

\[ K_1 = A_2 - B_2 + \frac{(B_1 - A_1)}{2\nu}[\nu(A_1 + B_1) + A_1 - B_1] + \frac{1 - \nu^2}{24\nu}, \]

\[ \prod_{j=1}^{p} (x + a_j) = \sum_{i=0}^{p} A_i x^{n-i}, \quad \prod_{j=1}^{q} (x + b_j) = \sum_{i=0}^{q} B_i x^{n-i}, \]

and the remaining \( K_j \) are polynomials in \( A_i, B_j \) independent of \( w \).

When \( \nu \geq 1 \), the contour \( L \) of (1.3) is equal to \( Z_0 \). In particular, if \( m = 0 \), \( G_m^m(z) = 0 \). A straightforward computation shows that \( L_i(ze^{i\pi(\mu - 1 - 2r)}) \) and \( G(ze^{i\pi(\mu - 2s)}) \) are also solutions of (1.4), \( r, s \) arbitrary integers. For a given value of \( \arg z \), there exists at least one pair of integers \( (r, s) \) such that

\[ |\arg z + \pi(\mu + 1 - 2r)| < \pi(\nu/2 + 1), \]

\[ |\arg z + \pi(\mu + 2 - 2s - 2h)| < \pi(\nu + \min(1, \nu/2)), \quad h = 1, \ldots, \nu. \]

It then follows from Theorem 2, that in a suitable sector, the \( q \) functions \( L_i(ze^{i\pi(\mu - 1 - 2r)}) \), \( j = 1, \ldots, p \), \( G(ze^{i\pi(\mu + 2 - 2s - 2h)}) \), \( h = 1, \ldots, \nu \), form a basis of solutions for (1.4).

**Theorem 3 (Meijer).** Under the conditions of (1.1), \( \nu \geq 1 \), if the sector

\[ \pi\left(\nu - \mu - 2 + \max\left[2r - \frac{3\nu}{2}, 2s - \min\left(1, \frac{\nu}{2}\right)\right]\right) < \arg z \]

\[ \arg z < \pi\left(\frac{\nu}{2} - \mu + \min\left[2r, \frac{\nu}{2} + 2s + \min\left(1, \frac{\nu}{2}\right)\right]\right) \]

is not empty, then there exist constants \( C_i(r, s), D_i(r, s) \) such that

(1.12)

\[ G_m^m(z) = \sum_{i=1}^{p} C_i(r, s)L_i(ze^{i\pi(\mu + 1 - 2r)}) + \sum_{h=1}^{s} D_h(r, s)G(ze^{i\pi(\mu + 2 - 2s - 2h)}). \]

Equation (1.12) will be referred to as the \( (r, s) \) expansion for \( G_m^m(z) \).

Once the values of \( C_i(r, s), D_i(r, s) \) have been determined for \( z \in S_{r,s} \), (1.12) remains valid for all values of \( \arg z \). However, it is useful for the asymptotic evaluation of \( G_m^m(z) \) only when the arguments of \( L_i \) and \( G \) functions which are actually present satisfy the argument restrictions of (1.8) and (1.9), respectively.

Thus, the only practical difficulty in using Theorem 3 to determine the behaviour of \( G_m^m(z) \) for \( z \) large lies in the determination of \( C_i(r, s), D_i(r, s) \) for any given value of \( \arg z \). This problem is discussed in Section 2.

If the complex conjugate of the \( (r, s) \) expansion, (1.12), is taken, treating \( z, a_j, b_j \) as real, one obtains the \( (\mu + 1 - r, \mu + 1 - \nu - s) \) expansion. This is particularly useful when \( \nu = 1 \) or 2.

**2. Coefficient Determination.** The practical problem of determining the \( C_i(r, s), D_i(r, s) \) for any given value of \( \arg z \) is simplified by noticing that for \( \nu \) fixed, it is sufficient to consider only certain diagonal sectors \( S_{r,s} \).

**Proposition 1.** For \( \nu \geq 1 \), let \( k_0 \) be the nonnegative integer such that
(2.1) \[\nu - 1 \leq 4k_0 \leq \nu + 2, \text{ or } 4k_0 - 2 \leq \nu \leq 4k_0 + 1.\]

Then
\[S_{r, r-k_0}: \pi(\nu - \mu + 2r - 3 - 2k_0) < \arg z < \pi(\nu/2 - \mu + 2r); \quad \nu \geq 2, k_0 \geq 1,\]
\[S_{r, r}: \pi(2r - \mu - \frac{3}{2}) < \arg z < \pi(2r - \mu + \frac{1}{2}); \quad \nu = 1, k_0 = 0,\]
\[S_{r, r-1}: \pi(2r - \mu - \frac{3}{2}) < \arg z < \pi(2r - \mu - \frac{3}{2}); \quad \nu = 1,\]
and the diagonal sectors,
\[S_{r, r}, S_{r, r-1}, \quad r = 0, \pm 1, \pm 2, \cdots,\]
\[S_{r, r-k_0}, \quad r = 0, \pm 1, \pm 2, \cdots,\]
completely cover the z-plane.

Thus, if the \((r, s)\) expansion is known, and \(S_{r, r}\) is one of the sectors in (2.3), it is sufficient to give recursion relations for the \((r \pm 1, s)\) and \((r, s \pm 1)\) expansions. These follow directly from the following \(L_i\) and \(G\) recursion relationships.

**Proposition 2.** Under the conditions of (1.1), \(\nu \geq k \geq 1\), let the constants \(C_i(k), D_i(k)\) be chosen such that
\[(2.4) \quad (-1)^{\nu+1}(2\pi i)^{\nu+i}\Gamma(\nu/2 - a - \nu)\Gamma(1 - a + \nu)\]
\[\Gamma(a - t)\Gamma(1 - a + t) - 1 + \sum_{h=1}^{\nu} D_i(k)e^{-i\pi 2h},\]
an identity which can be built up from the case \(k = 1\), by repeated use of
\[(2.5) \quad \Gamma(a_i - t)\Gamma(1 - a_i + t) = e^{i\pi 2a_i}e^{-i\pi 2t}\Gamma(a_i - t)\Gamma(1 - a_i + t) + (-2\pi i)e^{i\pi a_i}e^{-i\pi t}.\]

In particular,
\[C_i(k) = (-1)^{\nu+1}(2\pi i)^{\nu+i}\Gamma(\nu/2 - a - \nu)\Gamma(1 - a + \nu)\Gamma(a - t)\Gamma(1 - a + t)\]
\[\Gamma(a - t)\Gamma(1 - a + t) + (-2\pi i)e^{i\pi a_i}e^{-i\pi t},\]
\[(2.6) \quad D_i(k) = D_i(1) + (-2\pi i)\sum_{h=1}^{\nu} C_i(1)e^{i\pi(2h-1)a_i}, \quad 1 \leq h \leq k - 1,\]
\[= D_i(1), \quad k \leq h \leq \nu,\]
\[D_i(1) = (-1)^{\nu+1}e^{i\pi 2(\nu/2 - a_i)}G:\]

Then,
\[(2.7) \quad L_i(w) = e^{i\pi 2a_i}L_i(we^{-i\pi 2}) + (-2\pi i)e^{i\pi a_i}G(we^{-i\pi}),\]
\[(2.8) \quad G(w) = \sum_{i=1}^{p} C_i(k)L_i(we^{i\pi(1-2k)}) + \sum_{h=1}^{\nu} D_i(k)G(we^{-i\pi 2h}).\]

**Proof.** The existence of the expansion (2.4) follows from the partial fraction decomposition
\[
\prod_{j=1}^{r} \frac{(y - \beta_j)}{(y - \alpha_j)} = \sum_{\ell=0}^{\nu} c_{\ell,k} y^\ell + \sum_{h=0}^{\nu} d_{h,k} y^h, \\
(2.9)
\]

\[d_{r,k} = 1 \quad d_{0,k} = (-1)^{r} \prod_{j=1}^{r} \frac{\beta_j}{\alpha_j}, \quad \alpha_j \neq \alpha_n, \quad j \neq r, \quad 0 < k \leq \nu = q - p,
\]

with \(y = e^{-i\pi y}, \beta_j = e^{-i\pi \nu y},\) and \(\alpha_j = e^{-i\pi \nu y}.
\]

Multiplying (2.5), (2.4) by \(w^r \Gamma(b_0 - t)[\Gamma(q - t)]^{-1},\)

and integrating along a contour \(L,\) which separates the poles of \(\Gamma(b_0 - t)\) from those of \(\Gamma(q - t),\) we arrive at (2.7) and an expansion of \(G_{\nu,q}(we^{-i\pi r}) \equiv 0\) which reduces to (2.8), respectively. □

**Remark 1.** Equations (2.7) and (2.8) can also be written in the form

\[
L_\nu(w) = e^{-i\pi \nu y} L_{\nu}(we^{-i\pi y}) + (2\pi i) e^{-i\pi \nu y} G(we^{-i\pi y}), \\
(2.10)
\]

\[-D_\nu(1)G(we^{-i\pi y}) = \sum_{j=1}^{\nu} C_j(k) L_j(we^{-i\pi j}) + \sum_{h=1}^{\nu} D_{\nu-1}(k) G(we^{-i\pi (\nu - h)}),
\]

\[1 \leq k \leq \nu, \quad D_0(k) = -1.
\]

Using Proposition 2, the variables of \(L_\nu(w)\) and \(G(w)\) can be changed in a systematic fashion.

**Proposition 3.** Under the conditions (1.1), let \(k = r - s, 1 \leq k + 1 \leq \nu.\) Then the following recursion relations hold.

\[
(r, s) \rightarrow (r + 1, s)
\]

\[C_i(r + 1, s) = e^{i\pi 2s} C_i(r, s),
\]

\[D_h(r + 1, s) = D_h(r, s), \quad h \neq k + 1,
\]

\[= D_{h+1}(r, s) + (-2\pi i) \sum_{j=1}^{\nu} C_j(r, s) e^{i\pi j}, \quad h = k + 1,
\]

\[(r + 1, s) \rightarrow (r + 1, s + 1)
\]

\[C_i(r + 1, s + 1) = C_i(r + 1, s) + D_i(r + 1, s) C_i(k + 1),
\]

\[D_h(r + 1, s + 1) = D_i(r + 1, s) D_h(k + 1) + D_{h+1}(r + 1, s),
\]

\[1 \leq h \leq \nu - 1,
\]

\[(r, s) \rightarrow (r, s - 1)
\]

\[C_i(r, s - 1) = C_i(r, s) - D_i(r, s) C_i(k + 1),
\]

\[D_h(r, s - 1) = \frac{D_i(r, s)}{D_i(1)}, \quad h = 1,
\]

\[= D_{h-1}(r, s) - \frac{D_i(r, s)}{D_i(1)} D_{h-1}(k + 1), \quad 2 \leq h \leq \nu,
\]
(r, s - 1) \to (r - 1, s - 1)

\[ C_i(r - 1, s - 1) = e^{-i\pi 2a_i} C_i(r, s - 1), \]

\[ D_h(r - 1, s - 1) = D_h(r, s - 1), \quad h \neq k + 1, \]

\[ = D_{k+1}(r, s - 1) + (2\pi i) \sum_{i=1}^{p} C_i(r, s - 1)e^{-i\pi a_i}. \]

These recursion formulae are valid if \( k = k_0. \)

Proof. Equations (2.12) and (2.15) follow from (2.7) and (2.10), respectively, whereas Eqs. (2.13) and (2.14) follow when the \( h = 1 \) and \( h = v \) terms in the \((r + 1, s)\) or \((r, s)\) expansions are replaced by the expansions given in (2.8) and (2.11), respectively. □

Suppose that an \((s, s)\) expansion for \( G_{p,q}^n(z) \) is known, \( s \) arbitrary. Then \( k_0 \) applications of (2.12) yield the \((s + k_0, s)\) expansion for \( G_{p,q}^n(z) \), and Propositions 1, 3 imply the asymptotic behaviour of \( G_{p,q}^n(z) \) for all values of \( \arg z \) as \( z \to \infty \). Hence, the only remaining practical problem is to find a particular \((s, s)\) expansion.

Proposition 4. Under the conditions of (1.1), \( \nu \geq 1 \), there exist constants \( E_i, F_h \) and \( H_h \) such that

\[ \frac{\Gamma(a_N - t)\Gamma(1 - a_N + t)}{\Gamma_n(b_q - t)\Gamma_n(1 - b_q + t)} = \sum_{i=1}^{n} E_i e^{i\pi (\mu+1-2s)t} \Gamma(a_i - t)\Gamma(1 - a_i + t) \]

\[ + \sum_{h=1}^{s} F_h e^{i\pi (\mu+2-2s-2h)t}, \]

\( s = \text{max}(0, 1 + \mu - \nu). \) Then,

\[ (2.17) \quad E_i = C_i(s, s), \quad F_h = D_h(s, s). \]

In particular, if \( \mu \leq \nu - 1 \), (2.16) reduces to

\[ \frac{\Gamma(a_N - t)\Gamma(1 - a_N + t)}{\Gamma_n(b_q - t)\Gamma_n(1 - b_q + t)} = \sum_{i=1}^{n} C_i(0, 0)e^{i\pi (\mu+1)t} \Gamma(a_i - t)\Gamma(1 - a_i + t) \]

\[ + \sum_{h=1}^{1+\mu} D_h(0, 0)e^{i\pi (\mu+2-2h)t}, \]

\[ C_i(0, 0) = e^{-i\pi (\mu+1)a_i} \Gamma(1 + a_i - a_N)\Gamma(a_N - a_i) \Gamma_n(1 - b_q + a_i)\Gamma_n(b_q - a_i), \quad j = 1, \ldots, n. \]

If \( \mu \leq -1 \), the sum \( \sum_{h=1}^{1+\mu} \) in (2.18) also disappears.

Proof. We begin by noticing that (2.17) is a direct consequence of (2.16). For if (2.16) is multiplied by \( z^\nu \Gamma(b_q - t)\Gamma(a_P - t)\Gamma^{-1} \), and the resulting expansion is integrated over the contour \( L_+ \) of (1.3), one obtains the \((s, s)\) expansion of

\[ G_{p,q}^n(z) = \sum_{j=1}^{n} H_j G_{p,q}^0(z) e^{i\pi (\mu+2-2j)t} \]

\[ G_{p,q}^n(z). \]
First, assume that $\mu \leq \nu - 1$. Then (2.18) follows directly from the partial fraction decomposition

$$
(2.20) \quad \frac{\prod_{i=m+1}^{n} (y - \beta_i)}{\prod_{i=1}^{n} (y - \alpha_i)} = \sum_{i=1}^{n} \frac{e_i}{y - \alpha_i} + \sum_{h=0}^{n} f_h y^h,
$$

with $y = e^{-ivx^2}$, $\beta_i = e^{-ivx^2}$, and $\alpha_i = e^{-ivx^2}$. Similarly for the case, $\mu = \nu$, we prove by induction on $s$, $0 \leq s \leq 1 + \mu - \nu$, that there exist constants $e_{i,s}$, $f_{i,s}$, and $h_{i,s}$ such that

$$
(2.21) \quad \frac{\prod_{i=m+1}^{n} (y - \beta_i)}{\prod_{i=1}^{n} (y - \alpha_i)} = \left( \sum_{i=0}^{t-1} h_{i,s} y^i \right) \prod_{i=1}^{p} (y - \beta_i) = \sum_{i=1}^{n} \frac{e_{i,s} y^i}{y - \alpha_i} + \sum_{h=0}^{n} f_{h,s} y^h, \quad 0 \leq s \leq 1 + \mu - \nu.
$$

If $s = 0$, (2.21) reduces to (2.20). Assuming (2.21) valid for a particular value of $s \leq \mu - \nu$, we note that systematic application of

$$
(2.22) \quad 1/(y - a) = y/a(y - a) - 1/a,
$$

allows us to write $R(y)$, the left-hand side of (2.21), in the form

$$
(2.23) \quad R(y) = \sum_{i=1}^{p} \frac{e_{i,s} y^{i+1}}{\alpha_i} + y^s \left( f_{i,s} - \sum_{i=1}^{p} \frac{e_{i,s}}{\alpha_i} \right) + \sum_{h=0}^{n} f_{h,s} y^h.
$$

Finally, if the expansion (2.9) with $k = 1$ is multiplied by $y^h h_{i,s}$,

$$
(2.24) \quad d_{i,s} h_{i,s} = f_{i,s} - \sum_{i=1}^{p} \left( e_{i,s}/\alpha_i \right),
$$

and the resulting expression is subtracted from (2.23), one obtains (2.21) with $s$ replaced by $s + 1$. Equation (2.16) then follows from (2.21) with $s = 1 + \mu - \nu$, $y = e^{-ivx^2}$, $\beta_i = e^{-ivx^2}$, and $\alpha_i = e^{-ivx^2}$. $\Box$

**Remark 2.** A particular $(r, s)$ expansion may directly imply the asymptotic expansion of $G_{r,s}^{\alpha,\beta}(z)$ in a larger sector than $S_{r,s}$. For example, if $\nu \geq 1$, $\mu \leq -1$, it follows from Proposition 4 that

$$
(2.25) \quad G_{r,s}^{\alpha,\beta}(z) = \sum_{i=1}^{n} C_i(0, 0) L_i\left( z e^{i\pi(r+1)} \right),
$$

an expansion which, though valid for all values of arg $z$, directly implies the asymptotic behaviour of $G_{r,s}^{\alpha,\beta}(z)$ only when $-\pi(\nu/2 + \mu + 2) < \arg z < \pi(\nu/2 - \mu)$.

**Remark 3.** Once a particular $(r, s)$ expansion has been derived, the conditions (1.1) can be weakened by an appeal to analytic continuation.

**Remark 4.** The above results are easily transformed into results for the hypergeometric functions,

$$
(2.26) \quad \frac{\Gamma(\beta q)}{\Gamma(\alpha P)} \quad _pF_q \left( \begin{array}{c} \alpha P \\ \beta q \end{array} \left. \begin{array}{c} -z \\ 0, 1 - \beta q \end{array} \right| \Gamma(\beta q - t)[\Gamma(\alpha P - t)]^{-1} \right) \quad \text{and integrating the resultant}
$$

3. Generalizations. The above expansions are derived by multiplying certain identities by the function $z^j \Gamma(b_q - t)[\Gamma(a_p - t)]^{-1}$ and integrating the resultant.
identities over an appropriate contour. If these identities are multiplied by more general functions, the above expansions can be generalized.

Let $c_j, j = 1, \ldots, k; d_i, j = 1, \ldots, v; f_i, j = 1, \ldots, w$, be constants such that

\[(3.1) \quad c_i - b_u \neq 0 \text{ a positive integer}; \quad j = 1, \ldots, k; \quad u = 1, \ldots, q.\]

Then Theorem 3 can be generalized as follows.

**Theorem 4.** Let $a_i, b_i, c_i$ satisfy the conditions (1.1) and (3.1). If the expansion (1.12) is derivable from Propositions 2, 3 and 4, then

\[
G_{p+k+q+w}^{m, n+2k} \left( \begin{array}{c} c_k, a_p, d_v \\ b_q, f_w \end{array} \right) = \sum_{i=1}^{q} C_i(r, s) G_{p+k+q+w}^{r+1, s} \left( \begin{array}{c} c_k, a_p, a^{s^i} \\ b_q, f_w \end{array} \right) + \sum_{k=1}^{w} D_k(r, s) G_{p+k+q+w}^{r+k} \left( \begin{array}{c} c_k, a_p, d_v \\ b_q, f_w \end{array} \right),
\]

where $a^{s^i}$ denotes the sequence $a_1, \ldots, a_p$ with $a_i$ deleted, and $v, \mu, C_i(r, s), D_k(r, s)$ have the same values as in (1.12).

**Proof.** If the expansion (1.12) can be built up from Propositions 2, 3 and 4, the expansion (3.2) can be built up from appropriate generalizations of (2.7), (2.8) and (2.19). In particular, if (2.5) and (2.4) are multiplied by

\[
z^r \Gamma(b_q - t) \Gamma(1 - c_k + t) \Gamma(d_v - t) \Gamma(1 - f_w + t) \Gamma(1 - a_p + t)\]

and integrated over a contour $L^*_t$ which separates the poles of $\Gamma(b_q - t)$ from those of $\Gamma(1 - a_p + t)\Gamma(1 - c_k + t)$, one obtains

\[
L^*_t(z) = e^{i\pi 2a} L^*_t(ze^{-i\pi}) + (2\pi i) e^{i\pi 0} G^*(ze^{i\pi}),
\]

where $1 \leq u \leq v$, and

\[
G^*(z) = \sum_{i=1}^{p} C_i(u) L^*_i(ze^{i(1-2u)}) + \sum_{k=1}^{w} D_k(u) G^*(ze^{i(2k)}),
\]

Treating (2.16) similarly, one obtains (3.2) with $r = s$. \(\square\)

The significance of (3.2) lies in the fact that if $c_k, d_v$ and $f_w$ are "large", then in some restricted sense and in some restricted region,

\[
CG^*(z) \text{ behaves like } G(\Omega z),
\]

\[
CL^*_t(z) \text{ behaves like } L^*_t(\Omega z),
\]

\[
C = \frac{\Gamma(d_v) \Gamma(1 - f_w)}{\Gamma(1 - c_k)}, \quad \Omega = \prod_{j=1}^{v} d_j \prod_{i=1}^{k} (1 - c_i) \prod_{i=1}^{w} (1 - f_i).
\]

A special $k = 1, w = 0, v = 1$ case of Theorem 4 is discussed in [4].

