

# The Condition of Orthogonal Polynomials

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**Abstract.** An estimate is given for the condition number of the coordinate map associating to each polynomial its coefficients with respect to a system of orthogonal polynomials.

Let  $w(x) \geq 0$  be a weight function on the finite interval  $[a, b]$ , and  $\{p_k(x)\}_{k=0}^\infty$  the associated orthogonal polynomials. We consider the linear parametrization map  $M_n: \mathbf{R}^n \rightarrow \mathbf{P}_{n-1}$  which associates to each (real) vector  $u^T = [u_0, u_1, \dots, u_{n-1}] \in \mathbf{R}^n$  the (real) polynomial  $p(x) = \sum_{k=0}^{n-1} u_k p_k(x) \in \mathbf{P}_{n-1}$ . The object of this note is to estimate the condition

$$\text{cond}_\infty M_n = \|M_n\|_\infty \|M_n^{-1}\|_\infty$$

of the map  $M_n$ , the infinity norms in  $\mathbf{R}^n$  being defined by  $\|u\|_\infty = \max_{0 \leq k \leq n-1} |u_k|$ , and in  $\mathbf{P}_{n-1}$  by  $\|p\|_\infty = \max_{a \leq x \leq b} |p(x)|$ . Letting

$$\mu_0 = \int_a^b w(x) dx, \quad h_k = \int_a^b p_k^2(x) w(x) dx, \quad k = 0, 1, 2, \dots,$$

we show in fact that

$$(1) \quad \text{cond}_\infty M_n \leq \max_{0 \leq k \leq n-1} \left( \frac{\mu_0}{h_k} \right)^{1/2} \max_{a \leq x \leq b} \sum_{k=0}^{n-1} |p_k(x)|.$$

For Chebyshev polynomials  $p_k(x) = T_k(x)$  on  $[-1, 1]$ , e.g., this gives

$$\text{cond}_\infty M_n \leq 2^{1/2} n \quad (p_k = T_k),$$

while for Legendre polynomials  $p_k(x) = P_k(x)$  on  $[-1, 1]$  one gets

$$\text{cond}_\infty M_n \leq n(2n - 1)^{1/2} \quad (p_k = P_k).$$

In order to prove (1), we first observe that, for any  $u \in \mathbf{R}^n$ ,

$$\|M_n u\|_\infty = \left\| \sum_{k=0}^{n-1} u_k p_k(x) \right\|_\infty \leq \|u\|_\infty \max_{a \leq x \leq b} \sum_{k=0}^{n-1} |p_k(x)|,$$

so that

$$(2) \quad \|M_n\|_\infty \leq \max_{a \leq x \leq b} \sum_{k=0}^{n-1} |p_k(x)|.$$

On the other hand, if  $M_n^{-1} p = u$ , then, by orthogonality,

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$$u_k = \frac{1}{h_k} \int_a^b p(x) p_k(x) w(x) dx, \quad k = 0, 1, \dots, n-1.$$

Therefore, using the Schwarz inequality,

$$\begin{aligned} |u_k| &\leq \frac{1}{h_k} \int_a^b |p(x)| (w(x))^{1/2} \cdot |p_k(x)| (w(x))^{1/2} dx \\ &\leq \frac{1}{h_k} \left( \int_a^b p^2(x) w(x) dx \int_a^b p_k^2(x) w(x) dx \right)^{1/2} \\ &\leq \frac{1}{h_k} \left( \|p\|_\infty^2 \int_a^b w(x) dx \cdot h_k \right)^{1/2} = \|p\|_\infty (\mu_0/h_k)^{1/2}. \end{aligned}$$

It follows that, for all  $p \in \mathbf{P}_{n-1}$ ,

$$\|M_n^{-1} p\|_\infty \leq \|p\|_\infty \max_{0 \leq k \leq n-1} (\mu_0/h_k)^{1/2},$$

so that

$$(3) \quad \|M_n^{-1}\|_\infty \leq \max_{0 \leq k \leq n-1} (\mu_0/h_k)^{1/2}.$$

Combining (2) and (3) gives the desired result (1).

In terms of the orthonormal polynomials  $\pi_k(x) = h_k^{-1/2} p_k(x)$ , we may write (1) in the form

$$(1') \quad \text{cond}_\infty M_n \leq \max_{0 \leq k \leq n-1} (\mu_0/h_k)^{1/2} \max_{a \leq x \leq b} \sum_{k=0}^{n-1} h_k^{1/2} |\pi_k(x)|.$$

If we let  $h = \min_{0 \leq k \leq n-1} h_k$ , we see that the bound in (1') is larger than or equal to

$$(\mu_0/h)^{1/2} \max_{a \leq x \leq b} \sum_{k=0}^{n-1} h^{1/2} |\pi_k(x)| = \mu_0^{1/2} \max_{a \leq x \leq b} \sum_{k=0}^{n-1} |\pi_k(x)|,$$

so that, among all possible normalizations, the one with  $h_0 = h_1 = \dots = h_{n-1}$  gives the best bound in (1).

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