Remark on a Paper by Huddleston

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Abstract. Using a function-theoretic approach, we obtain, in a quite simple way, linear relations between the values of a function and its first derivatives at \( n \) abscissa points \( x_1, \ldots, x_n \). The derivation of these formulae in a recent paper by Huddleston was rather cumbersome. Possible generalizations are indicated.

1. In a recent paper, Huddleston [1] gave some relations between the values of a function and its first derivatives at \( n \) abscissa points. Huddleston's derivation is, to speak with his own words, "an exercise in drudgery". Using a function theoretic approach, we give a new proof of the results in [1] which is both simple and lucid and, in addition, indicates how one may obtain more general relations by the same method.

2. Let \( C \) be a simple closed rectifiable positively oriented curve in the complex plane. Let \( x_1 < x_2 < \cdots < x_n \) be \( n \) points on the real axis which lie in the interior of \( C \), and let

\[
w(z) = (z - x_1)(z - x_2) \cdots (z - x_n).
\]

For functions \( f(z) \), holomorphic in a domain \( G \) which contains \( C \), consider the linear functional

\[
L(f) = \frac{1}{2\pi i} \int_{C} \frac{f(z)}{w'(z)} \, dz.
\]

Clearly, \( L(f) \) vanishes if \( f(z) \) is a polynomial \( P_{2n-2}(z) \) of degree less than or equal to \( 2n - 2 \). From the Taylor series

\[
f(z) = f(x_i) + f'(x_i)(z - x_i) + \cdots
\]

and

\[
w^2(z) = w''(x_i)(z - x_i)^2 + w'(x_i)w''(x_i)(z - x_i)^3 + \cdots,
\]

we get

\[
\text{res}_{z = x_i} \frac{f(z)}{w^2(z)} = \frac{1}{w^2(x_i)} f'(x_i) - \frac{w''(x_i)}{w^3(x_i)} f(x_i),
\]

and the residue theorem gives

\[
L(f) = \sum_{i=1}^{n} \frac{1}{w^2(x_i)} f'(x_i) - \frac{w''(x_i)}{w^3(x_i)} f(x_i).
\]

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For \( f(z) = P_{2n-2}(z) \), we obtain Huddleston’s formula

\[
\sum_{\nu=1}^{n} \frac{1}{w^\nu(x_\nu)} P'_{2n-2}(x_\nu) - \frac{w'/(x_\nu)}{w^\nu(x_\nu)} P_{2n-2}(x_\nu) = 0.
\]

3. Using the fact that \( L(f) \) is equal to the divided difference with coalescent knots \([x_1, x_1, x_2, x_3, \ldots, x_n] \) (see [2, p. 199]), we get in the case that \( f(z) \) is real for real \( z \):

\[
L(f) = \frac{f^{(2n-1)}(\xi)}{(2n - 1)!}, \quad \xi \in (x_1, x_n)
\]
(see [2, p. 13]). Huddleston’s formula (5.1), (5.2) is a consequence of (1), (2) and (3).

4. In the case of equidistant knots, e.g. \( x_\nu = \nu, \nu = 1(1)n \), we arrive at

\[
\sum_{\nu=1}^{n} \left( \frac{n - 1}{\nu - 1} \right)^2 \left[ f'(\nu) - \sum_{\mu=1; \mu \neq \nu}^{n} \frac{2}{\nu - \mu} f(\nu) \right] = \frac{[(n - 1)!]^2}{(2n - 1)!} f^{(2n-1)}(\xi).
\]

5. Obviously, our method may be generalized to obtain similar relations for other Hermite data.

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