Remark on a Paper by Huddleston

By Gerhard Merz

Abstract. Using a function-theoretic approach, we obtain, in a quite simple way, linear relations between the values of a function and its first derivatives at n abscissa points $x_1, \ldots, x_n$. The derivation of these formulae in a recent paper by Huddleston was rather cumbersome. Possible generalizations are indicated.

1. In a recent paper, Huddleston [1] gave some relations between the values of a function and its first derivatives at n abscissa points. Huddleston's derivation is, to speak with his own words, "an exercise in drudgery". Using a function theoretic approach, we give a new proof of the results in [1] which is both simple and lucid and, in addition, indicates how one may obtain more general relations by the same method.

2. Let $C$ be a simple closed rectifiable positively oriented curve in the complex plane. Let $x_1 < x_2 < \cdots < x_n$ be n points on the real axis which lie in the interior of $C$, and let

$$w(z) = (z - x_1)(z - x_2) \cdots (z - x_n).$$

For functions $f(z)$, holomorphic in a domain $G$ which contains $C$, consider the linear functional

$$(1)\ L(f) = \frac{1}{2\pi i} \int_C \frac{f(z)}{w^2(z)} \frac{dw}{dz} dz.$$ 

Clearly, $L(f)$ vanishes if $f(z)$ is a polynomial $P_{2n-2}(z)$ of degree less than or equal to $2n - 2$. From the Taylor series

$$f(z) = f(x_s) + f'(x_s)(z - x_s) + \cdots$$

and

$$w^2(z) = w'^2(x_s)(z - x_s)^2 + w'(x_s)w''(x_s)(z - x_s)^3 + \cdots,$$

we get

$$\text{res}_{z=x_s} \frac{f(z)}{w^2(z)} = \frac{1}{w'^2(x_s)} f'(x_s) - \frac{w''(x_s)}{w'^3(x_s)} f(x_s),$$

and the residue theorem gives

$$L(f) = \sum_{s=1}^n \frac{1}{w'^2(x_s)} f'(x_s) - \frac{w''(x_s)}{w'^3(x_s)} f(x_s).$$

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For \( f(z) = P_{2n-2}(z) \), we obtain Huddleston's formula

\[
\sum_{r=1}^{n} \frac{1}{w^{2r}(x_r)} P'_{2n-2}(x_r) - \frac{w^{r'}(x_r)}{w^{2r}(x_r)} P_{2n-2}(x_r) = 0.
\]

3. Using the fact that \( L(f) \) is equal to the divided difference with coalescent knots \([x_1, x_2, x_3, \ldots, x_n, x_n] \) (see [2, p. 199]), we get in the case that \( f(z) \) is real for real \( z \):

\[
L(f) = \frac{f^{(2n-1)}(\xi)}{(2n - 1)!}, \quad \xi \in (x_1, x_n)
\]

(see [2, p. 13]). Huddleston's formula (5.1), (5.2) is a consequence of (1), (2) and (3).

4. In the case of equidistant knots, e.g. \( x_r = v, v = 1(1)n \), we arrive at

\[
\sum_{r=1}^{n} \binom{n-1}{n-1} f'(v) - \sum_{\mu=1, \mu \neq r}^{n-2} \frac{2}{v - \mu} f(v) = \frac{[(n - 1)!]^2}{(2n - 1)!} f^{(2n-1)}(\xi).
\]

5. Obviously, our method may be generalized to obtain similar relations for other Hermite data.

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