How Slowly Can Quadrature Formulas Converge?

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Abstract. Let \( \{Q_n\}_{n=1}^\infty \) denote a sequence of quadrature formulas, \( Q_n(f) = \sum_{j=1}^{k_n} w_j^{(n)} f(x_j^{(n)}) \), such that \( Q_n(f) \to \int_0^1 f(x) \, dx \) for all \( f \in C[0, 1] \). Let \( 0 < \epsilon < \frac{1}{2} \) and a sequence \( \{a_n\}_{n=1}^\infty \) be given, where \( a_1 \geq a_2 \geq a_3 \geq \cdots \), and where \( a_n \to 0 \) as \( n \to \infty \). Then there exists a function \( f \in C[0, 1] \) and a sequence \( \{n_k\}_{k=1}^\infty \) such that \( |f(x)| \leq 2a_k/(1 - 4\epsilon) \), and such that
\[
\int_0^1 f(x) \, dx - Q_n(f) = a_k, \quad k = 1, 2, 3, \ldots.
\]

1. Introduction and Statement of Results. We consider a sequence of quadrature formulas \( \{Q_n\}_{n=1}^\infty \) defined by
\[
Q_n f = \sum_{j=1}^{k_n} w_j^{(n)} f(x_j^{(n)})
\]
where \( \{k_n\}_{n=1}^\infty \) is a sequence of (increasing) positive integers and \( 0 \leq x_j^{(n)} \leq 1 \) for all \( n \) and \( j \). The quadrature formulas, we assume, are such that
\[
\lim_{n \to \infty} Q_n f = \int_0^1 f(x) \, dx
\]
for all functions \( f \) that are continuous on the closed interval \([0, 1]\); that is, for all \( f \) in \( C[0, 1] \). For example, the Gaussian quadrature formulas and the well-known trapezoidal formulas have these properties.

In this paper, we show that no matter what the sequence \( \{Q_n\}_{n=1}^\infty \) defined by (1.1) and (1.2) is, there is a function \( f \) in \( C[0, 1] \) for which \( \|Q_n f\|_{n=1}^\infty \) converges to \( I f \) very slowly. That is, the assumption of continuity is not enough to insure the rapid convergence of any quadrature scheme. More precisely, our main result is the following:

**Theorem 1.** Let a sequence of quadrature formulas \( \{Q_n\}_{n=1}^\infty \) defined by (1.1) and satisfying (1.2) for all \( f \) in \( C[0, 1] \) be given, and let \( \{a_n\} \) be any sequence of numbers such that
\[
\lim_{n \to \infty} a_n = 0,
\]
and
\[
S = \sum_{n=1}^{\infty} |a_n - a_{n+1}| < \infty.
\]

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Corresponding to any \( \epsilon \) such that \( 0 < \epsilon < \frac{1}{4} \), there exists a function \( f \) that is continuous and bounded by \( 2S/(1 - 4\epsilon) \) on the interval \([0, 1]\), and an increasing sequence of positive integers \( \{n_k\}_{k=1}^\infty \) such that

\[
|f - Q_{n_k}| = a_k \quad \text{for} \quad k = 1, 2, \cdots .
\]

We also have

**Corollary 2.** For every integer \( N > 0 \), there exists a polynomial \( P \) bounded on the interval \([0, 1]\) by \( 2S/(1 - 4\epsilon) \) such that

\[
|P - Q_{n_k}P| = a_k \quad \text{for} \quad k = 1, 2, \cdots , N.
\]

**Remark 3.** We remark that if the sequence \( \{a_k\}_{k=1}^\infty \) is a monotonically increasing or decreasing sequence of real numbers, then \( S = |a_1| \).

In the following section, we will construct the function \( f \) of Theorem 1 as a uniformly convergent sum of linear spline functions (broken linear functions) on the interval \([0, 1]\). This constructive proof of Theorem 1 leads to an elementary proof of Corollary 2. In the final section, we consider a particular sequence of quadrature formulas and show the easier calculation involved in this case.

Some historical remarks are in order. Let \( P^n \) denote the set of polynomials of degree \( n - 1 \) in \( x, n = 1, 2, 3, \cdots \). In 1938, Bernstein [1] proved that, given any sequence of positive numbers \( a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_n \geq \cdots \geq 0 \), where \( a_n \to 0 \) as \( n \to \infty \), there exists a function \( f \in C[0, 1] \) such that

\[
\inf_{p \in P^n} \left\{ \sup_{x \in (0, 1)} |f(x) - p(x)| \right\} = a_n, \quad n = 1, 2, 3, \cdots .
\]

This result has since been cast in the terminology of best approximation in normed linear spaces (Timan [2, p. 40]). From the point of view of practical applications, Bernstein's theorem tells us that there are continuous functions defined on the interval \([0, 1]\) which cannot be approximated to a desired accuracy by polynomials.

Our paper extends a recent result of Chui [5] who proved that given \( \{a_n\}_{n=1}^\infty \), a sequence of positive numbers which converges monotonically to zero, there exists a Riemann integrable function \( f \) such that

\[
\left| \int_0^1 f(x) \, dx - \frac{1}{n} \sum_{k=1}^n f\left(\frac{k}{n}\right) \right| \geq a_n \quad \text{for} \quad n = 1, 2, \cdots .
\]

In 1933 Pólya [3] constructed an analytic function for which the Newton-Cotes quadrature scheme diverges. Pólya's result was preceded by an interesting asymptotic estimate of the remainder of Newton-Cotes quadrature due to Ouspensky [6]; in [6], Ouspensky concluded that the Newton-Cotes formulas were devoid of any practical value. Thus, while proofs of convergence of quadrature schemes applied to continuous functions are very interesting (see Pólya [3] and Espinoza-Maldonado and Byrne [4]), the point of our paper in the spirit of Ouspensky is that when quadrature formulas are applied to continuous functions, the results may be devoid of any practical value.

2. **Constructive Proofs.** Let \( \Pi_i \) denote the set of evaluation points of \( Q_{n_i} \). The function \( f \) will be of the form
(2.1) \[ f(x) = \sum_{i=1}^{\infty} \alpha_i s_i(x) \]

where the \( s_i \) are to be constructed and the \( \alpha_i \) are to be determined.

Choice of \( n_i \) and Construction of \( s_i(x) \). We fix \( \epsilon \) such that \( 0 < \epsilon < \frac{1}{4} \). Then, by (1.2), there is an integer \( n_1 \) such that

(2.2) \[ |Q_{n_1} - I(1)| < \epsilon/2. \]

We now define \( s_1 \) to be the linear spline function whose graph has the vertices

(i) \( s_1(x) = 1 \) for all \( x \in \Pi_1 \), and
(ii) \( s_1(x) = 0 \) for all \( x \) midway between consecutive points of \( \Pi_1 \), and also for \( x \in [0, 1] - \Pi_1 \).

Thus, if \( \Pi_1 = \{z_1, z_2, z_3\} \), we would have the following graph:

Since \( s_1(x) = 1 \) on \( \Pi_1 \), we have, by (2.2),

(2.3) \[ Q_{n_1} s_1 = 1 + \eta_{11} \quad \text{where } |\eta_{11}| < \epsilon/2. \]

Also, clearly,

(2.4) \[ \int_0^1 s_1(x) \, dx = \frac{1}{2}. \]

We now proceed to the \( p \)th stage and suppose we have picked \( n_1, n_2, \cdots, n_{p-1} \) so that

(2.5) \[ \int_0^1 s_i(x) \, dx = \frac{1}{2} \quad \text{for } i = 1, 2, \cdots, p - 1. \]

Let \( \delta_p = \epsilon/p2^p \). We cover the union of the \( \Pi_i \) \( (i = 1, 2, \cdots, p - 1) \) with a finite union of open (and, possibly, half-open) intervals of total length \( \delta_p/3 \). We call this union \( C_p \) and cover its closure with another union of open (and half-open) intervals, \( B_p \), this time of total length \( 2\delta_p/3 \).

We now define a preliminary function \( T_p \) to be continuous on the interval \([0, 1]\) and such that

(i) \( T_p(x) = 0 \) for \( x \) in \( C_p \),
(ii) \( T_p(x) = 1 \) for \( x \) in \([0, 1] - B_p \), and
(iii) \( T_p(x) \) is linear elsewhere.

Thus, we might have the graph
where the union of the \( \Pi_i \) is \( \{z_1, z_2, z_3\} \). Then, from the definition of \( T_p \), we clearly have

\[
\int_0^1 (1 - T_p(x)) \, dx \leq 2\delta_p / 3.
\]

By (1.2), (2.5) and (2.6) there is an integer \( n_p \) such that

\[
Q_{n_p}s_j = 1 + \eta_{p,j} \quad \text{for } j = 1, 2, \ldots, p - 1 \quad \text{where } |\eta_{p,j}| < \delta_p, \quad \text{and}
\]

\[
Q_{n_p}T_p = 1 + \eta_{pp} \quad \text{where } |\eta_{pp}| < \delta_p.
\]

Finally, we define \( s_p(x) \) to be the linear spline function which is identical to \( T_p(x) \) on the closure of the set \( B_p - C_p \) and whose graph has the following additional vertices:

(i) \( s_p(x) = 1 \) for \( x \in \Pi_p \cap ([0, 1] - B_p) \),

(ii) \( s_p(x) = 0 \) for \( x \in \bigcup_{i=1}^{p-1} \Pi_i \), \( x \in [0, 1] - \Pi_p \) and \( x \in \Pi_p \cap C_p \),

(iii) \( s_p(x) = 0 \) for all \( x \) halfway between two consecutive “one” vertices so far defined, and,

(iv) \( s_p(x) = 1 \) for all \( x \) halfway between two consecutive “zero” vertices defined in (ii).

Then, \( s_p(x) \) has the following properties:

\[
Q_{n_p}s_p = Q_{n_p}T_p \quad \text{since } s_p = T_p \text{ on } \Pi_p.
\]

\[
\int_0^1 s_p(x) \, dx = \frac{1}{2}.
\]

\[
Q_{n_p}s_p = 0 \quad \text{for } i = 1, 2, \ldots, p - 1 \quad \text{by (ii)}.
\]

In this manner, we pick the subsequence \( \{n_p\}_{i=1}^{\infty} \) and the functions \( \{s_i\}_{i=1}^{\infty} \).

**Determination of \( \alpha_i \).** We can now apply the linear functional \( 1 - Q_{n_p} \) to the function \( f \) of (2.1), and, at the same time, impose the condition (1.5). We then obtain, using (2.6)–(2.10), the following infinite system of linear equations:

\[
\frac{1}{2} \sum_{i=1}^{\infty} \alpha_i - \left( \sum_{i=1}^{p-1} \alpha_i \left( \frac{1}{2} + \eta_{pi} \right) + \alpha_p \left( 1 + \eta_{pp} \right) \right) = a_p \quad \text{for } p = 1, 2, \ldots.
\]

If we subtract two such consecutive equations, we obtain the recurrence relation

\[
\alpha_{p+1} \left( 1 + \eta_{p+1,p+1} \right) = a_p - a_{p+1} + \alpha_p \left( \frac{1}{2} + \eta_{pp} - \eta_{p+1,p} \right) + \sum_{j=1}^{p-1} \alpha_j \left( \eta_{pj} - \eta_{p+1,j} \right).
\]
Setting $\alpha_i = 0$, we can use (2.12) to solve for all the $\alpha_i$.

It should be noted here that the passage from (2.11) to (2.12) is so far only a formal one, since the infinite series on the left-hand side of (2.11) has not yet been shown to converge. This will be shown in the next part of the proof.

$f$ is Continuous. In order to complete the proof of Theorem 1 we need only show that the $\alpha_i$, as determined above, make the function $f$ of (2.1) continuous on the interval [0, 1]. Thus, we must show that the infinite series (2.1) converges uniformly on the interval [0, 1]. Since $|s_i(x)| \leq 1$ for all $x$ and for every integer $i$, we need only show that

$$\sum_{p=1}^{\infty} |\alpha_p| < \infty.$$  

Indeed, from (2.12) and the bounds on $\eta_{pi}$ given in (2.7),

$$|\alpha_{p+1}| \leq |a_p - a_{p+1}| + \frac{1}{2} |\alpha_p| + \frac{\epsilon}{2^{p-1}} \sum_{i=1}^{p+1} |\alpha_i|.$$  

We now sum each side of (2.14) from 1 to $N$, where $N$ is an arbitrary but fixed positive integer. Replacing the sum $\sum_{i=1}^{N+1} |\alpha_i|$ on the right-hand side by $\sum_{i=1}^{N+1} |\alpha_i|$ and adding $\frac{1}{2} |\alpha_{N+1}|$ to the right-hand side, we obtain

$$\sum_{p=1}^{N} |\alpha_{p+1}| \leq \sum_{p=1}^{N} |a_p - a_{p+1}| + \frac{1}{2} \sum_{p=1}^{N} |\alpha_p| + \epsilon \sum_{p=1}^{N} \frac{1}{2^{p-1}} \sum_{i=1}^{N+1} |\alpha_i|.$$  

Transposing and simplifying gives

$$\sum_{p=1}^{N} |\alpha_{p+1}| \left(1 - \frac{1}{2} - 2\epsilon \left(1 - \frac{1}{2^N}\right)\right) \leq \sum_{p=1}^{N} |a_p - a_{p+1}| < \infty \quad \text{by (1.4)}.$$  

Now letting $N \to \infty$ on the left, we obtain

$$\sum_{p=1}^{\infty} |\alpha_{p+1}| \leq \frac{2}{1 - 4\epsilon} \sum_{p=1}^{\infty} |a_{p+1} - a_p|.$$  

This completes the proof of Theorem 1.

Proof of Corollary 2. The proof of Corollary 2 is similar to that of Theorem 1, with one exception. Here we also need to use Weierstrass' approximation theorem, which enables us to approximate each continuous function $s_p$ that we constructed in the proof of Theorem 1 by a polynomial $q_p$, such that $\max_{x \in [0, 1]} |s_p(x) - q_p(x)| < \delta_p/2$ for $p = 1, 2, \ldots, N$. We omit the details of this proof.

3. A Special Case. We consider the following sequence of midpoint quadrature formulas which certainly satisfy (1.2):

$$Q_{n/2} = \frac{1}{2^{n-1}} \left[ f\left( \frac{1}{2^n} \right) + f\left( \frac{3}{2^n} \right) + f\left( \frac{5}{2^n} \right) + \cdots + f\left( \frac{2^n - 1}{2^n} \right) \right].$$  

In this case there are no repeating evaluation points, and we can therefore choose $n_i = i$, and define $s_i$ to be the linear spline whose graph has the vertices

(i) $s_i(x) = 1$ for $x \in \Pi_i$, and
(ii) $s_i(x) = 0$ for $x \in \bigcup_{j=1}^{i-1} \Pi_i$ and also for $x = 0, 1$.  

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Then, as in (2.9),

\[ (3.2) \int_0^1 s_i(x) \, dx = \frac{1}{2} \quad \text{for all } i. \]

Instead of (2.6) and (2.7), we have the exact relations

\[ (3.3) Q_j s_i = \frac{1}{2} \quad \text{if } j < i, \]

\[ = 1 \quad \text{if } j = i, \]

\[ = 0 \quad \text{if } j > i. \]

Instead of (2.11), we have

\[ (3.4) \frac{1}{2} \sum_{i=1}^\infty \alpha_i - \frac{1}{2} \sum_{i=1}^{p-1} \alpha_i - \alpha_p = a_p \quad \text{for } p = 1, 2, \ldots. \]

Subtracting two such consecutive equations gives the recurrence relation

\[ (3.5) a_{p+1} = a_p - a_{p+1} + \frac{1}{2} \alpha_p, \]

which can be solved explicitly in terms of \( \alpha_1 \) to give

\[ (3.6) \alpha_{p+1} = a_p - a_{p+1} + \frac{1}{2} (a_{p-1} - a_p) \]

\[ + \frac{1}{2^3} (a_{p-2} - a_{p-1}) + \cdots + \frac{1}{2^{p-1}} (a_1 - a_2) + \frac{1}{2^p} \alpha_1. \]

The function \( f \) of the form (2.1) is easily seen to be continuous this time, since

\[ (3.7) \sum_{p=1}^\infty |\alpha_p| \leq 2 \sum_{p=1}^\infty |a_p - a_{p+1}| + 2 |\alpha_1|. \]

If we take \( \alpha_1 = 0 \), we note that

\[ (3.8) |f(x)| \leq 2 \sum_{p=1}^\infty |a_{p+1} - a_p|. \]