On the General Hermite Cardinal Interpolation

By R. Kress

Abstract. A sequence of interpolation series is given which generalizes Whittaker's cardinal function to the case of Hermite interpolation. By integrating the interpolation series, a sequence of new quadrature formulae for \( \int_{\infty}^{\infty} f(x) \, dx \) is obtained. Derivative-free remainders are stated for these interpolation and quadrature formulae.

Given a function \( f : \mathbb{R} \to \mathbb{C} \) and a real number \( h > 0 \), the series

\[
T_h(f)(z) := \frac{h}{\pi} \sum_{m=-\infty}^{\infty} \frac{(-1)^m f(mh)}{z - mh} \sin \frac{\pi}{h} z
\]

is called the cardinal series of the function \( f \) with respect to the interval \( h \). If the series converges, its sum \( T_h(f) \) is called the cardinal function or cardinal interpolation of the function \( f \). Obviously,

\[
T_h(f)(mh) = f(mh), \quad m = 0, \pm 1, \pm 2, \cdots
\]

holds. In the case when \( f : B \to \mathbb{C} \) is analytic in a strip \( B := \mathbb{R} \times [-a, a] \subseteq \mathbb{C}, a > 0, \) and satisfies certain conditions at infinity, a derivative-free remainder for this cardinal interpolation was independently found by Kress [2] and McNamee, Stenger and Whitney [6].

In the present paper, we generalize the cardinal interpolation and give a sequence of Hermite cardinal interpolations \( T_{p,h}(f) \), \( p = 0, 1, 2, \cdots \), with

\[
T_{p,h}^{(q)}(f)(mh) = f^{(q)}(mh), \quad q = 0, 1, \cdots, p, \quad m = 0, \pm 1, \pm 2, \cdots
\]

The usual cardinal interpolation is included as the particular case \( p = 0 \).

In Section 1, we give the explicit form of \( T_{p,h}(f) \) and state a derivative-free remainder. Making use of this remainder, we describe a class of functions for which \( T_{p,h}(f) = f \).

In Section 2, we apply the general cardinal functions to derive a sequence \( I_{p,h}(f) \), \( p = 0, 2, 4, \cdots \), of integration formulae for infinite integrals involving the derivatives \( f^{(q)}(mh), q = 0, 2, \cdots, p, \quad m = 0, \pm 1, \pm 2, \cdots \), which may be regarded as generalizations of the trapezoidal rule. The remainder, given by Goodwin [1], Martensen [4] and McNamee [5] for the trapezoidal rule, is extended to the quadrature formulae \( I_{p,h}(f) \).

Received December 27, 1971.

AMS 1970 subject classifications. Primary 30A80, 41A05, 41A55.

Key words and phrases. Cardinal function, Hermite interpolation, quadrature formulae, analytic functions, remainders, error bounds.

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The general cardinal interpolation developed in this paper is closely related to the general Hermite trigonometric interpolation of periodic functions [3].

1. Interpolation. Let \( p \geq 0 \) be an integer and let \( h > 0 \) be real. Define \( p + 1 \) entire functions \( t_q, q = 0, 1, \ldots, p, \) by

\[
t_q(z) = \frac{z^q}{q!} \left( \frac{\sin(\pi/h)z}{(\pi/h)z} \right)^{p+1} \sum_{r=0, r \text{ even}}^p a_r(h) \left( \frac{\pi}{h} z \right)^r, \quad z \in \mathbb{C},
\]

where the \( a_r(h) \) are the coefficients of the Laurent expansion

\[
\frac{1}{\sin^{p+1} z} = \sum_{r=0}^\infty a_r(\pi/h) z^{-r}, \quad 0 < |z| < \pi.
\]

To avoid indexing difficulties, we do not indicate the dependence of the \( t_q \) on \( p \) and \( h \).

**Lemma 1.1.** For every \( r = 0, 1, \ldots, p \), the functions \( t_q, q = 0, 1, \ldots, p, \) satisfy

\[
t_q^{(r)}(0) = \delta_q^r,
\]

\[
t_q^{(r)}(mh) = 0, \quad m = \pm 1, \pm 2, \ldots.
\]

**Proof.** From (1.1) and (1.2), we obtain

\[
t_q(z) = z^q/q! + z^{q+1} u_q(z), \quad q = 0, 1, \ldots, p,
\]

with certain entire functions \( u_q \), and (1.3) immediately follows. The relation (1.4) trivially holds.

**Definition 1.1.** Let \( p \geq 0 \) be an integer and let \( h > 0 \) be real. Given a function \( f: \mathbb{R} \rightarrow \mathbb{C}, f \in C^p(\mathbb{R}) \), the \( p \)th cardinal series of \( f \) with respect to the interval \( h \) is defined by

\[
T_{p,h}(f)(z) := \sum_{m=-\infty}^{\infty} \sum_{q=0}^p f^{(q)}(mh) t_q(z - mh), \quad z \in \mathbb{C}.
\]

If the series converges, its sum \( T_{p,h}(f) \) is called the \( p \)th cardinal function of \( f \).

**Lemma 1.1 implies**

**Theorem 1.1.** The \( p \)th cardinal function \( T_{p,h}(f) \) is a Hermite interpolation of the function \( f \) with equidistant interpolation points

\[
T_{p,h}(f)(mh) = f^{(q)}(mh), \quad q = 0, 1, \ldots, p, \quad m = 0, \pm 1, \ldots.
\]

The first cardinal series is listed below.

\[
T_{0,h}(f)(z) = \frac{h}{\pi} \sum_{m=-\infty}^{\infty} \frac{f(mh)}{z - mh} \sin \frac{\pi}{h} z,
\]

\[
T_{1,h}(f)(z) = \frac{(h/\pi)^2}{\pi} \sum_{m=-\infty}^{\infty} \left\{ \frac{f(mh)}{(z - mh)^2} + \frac{f'(mh)}{z - mh} \right\} \sin^2 \frac{\pi}{h} z,
\]

\[
T_{2,h}(f)(z) = \frac{(h/\pi)^3}{\pi} \sum_{m=-\infty}^{\infty} \frac{(-1)^m f(mh)}{(z - mh)^3} - \frac{1}{2} \frac{(h/\pi)^2}{(z - mh)^2} \frac{f(mh)}{z - mh} + \frac{f'(mh)}{z - mh} \sin^2 \frac{\pi}{h} z.
\]

In the case when the function \( f \) is analytic in a strip \( B := \mathbb{R} \times [-a, a] \subset \mathbb{C}, a > 0 \), we shall give a sufficient condition on the convergence of the \( p \)th cardinal
series of $f$ and shall obtain a representation of the remainder

\[(1.7)\]
\[R_{p,k}(f) := f - T_{p,k}(f).\]

**Lemma 1.2.** Let the function $f$ be analytic in the strip $B := \mathbb{R} \times [-a, a] \subset \mathbb{C}$, $a > 0$. Then

\[
\chi_{n}(z)/f(z) - \sum_{m=-n}^{n} \sum_{q=0}^{p} f^{(q)}(mh)t_{q}(z - mh) = \frac{1}{2\pi i} \sin^{p+1}(\pi/h)z \int_{c_{n}} \frac{f(\zeta) \, d\zeta}{(\zeta - z) \sin^{p+1}(\pi/h)\zeta}, \quad z \in C_{n},
\]

where $C_{n}$ denotes the boundary of a rectangle $B_{n} := [-a + (n + \frac{1}{2})h, a + (n + \frac{1}{2})h] \times [-a, a] \subset B$ and where $\chi_{n}$ denotes the characteristic function of $B_{n}$ with $\chi_{n}(z) = 1$, $z \in B_{n}$ and $\chi_{n}(z) = 0$, $z \not\in B_{n}$.

**Proof.** The function $F: B_{n} \to \mathbb{C}$, defined by

\[(1.9)\]
\[F(z) := \frac{1}{\sin^{p+1}(\pi/h)z} \left( f(z) - \sum_{m=-n}^{n} \sum_{q=0}^{p} f^{(q)}(mh)t_{q}(z - mh) \right), \quad z \in B_{n},
\]
is analytic. Hence, by Cauchy's theorem,

\[(1.10)\]
\[\chi_{n}(z)F(z) = \frac{1}{2\pi i} \int_{c_{n}} \frac{F(\zeta) \, d\zeta}{\zeta - z}, \quad z \in C_{n}.
\]

Using the identities

\[
\int_{c_{n}} \frac{d\zeta}{(\zeta - z)(\zeta - mh)^{q+1}} = \frac{-2\pi i}{(z - mh)^{q+1}} (1 - \chi_{n}(z)), \quad z \in C_{n},
\]

$q = 0, 1, \ldots, p$, $m = 0, \pm 1, \ldots, \pm n$, we substitute (1.9) into (1.10) and obtain (1.8).

**Theorem 1.2.** Let the function $f$ be analytic and bounded in the strip $B := \mathbb{R} \times [-a, a] \subset \mathbb{C}$, $a > 0$, and let

\[(1.11)\]
\[\int_{-\infty}^{\infty} |f(z)|^{2} \, ds < \infty, \quad \int_{-\infty}^{\infty} |f(z)|^{2} \, ds < \infty.
\]

Then, for arbitrary $p \geq 0$ and $h > 0$, the $p$th cardinal series of $f$ with respect to the interval $h$ is locally uniformly convergent for all $x \in \mathbb{R}$ and the remainder (1.7) is given by

\[
R_{p,k}(f)(x) = \frac{1}{2\pi i} \sin^{p+1}(\pi/h) \frac{x}{h} \int_{-\infty}^{\infty} f(\zeta) \frac{d\zeta}{(\zeta - x) \sin^{p+1}(\pi/h)\zeta} - \frac{f(\zeta)}{(\zeta - x) \sin^{p+1}(\pi/h)\zeta}
\]

\[(1.12)\]
\[x \in \mathbb{R},
\]
and bounded by

\[
|R_{p,k}(f)(x)| \leq \frac{1}{2(\pi a)^{1/2}} \left( \frac{\sin(\pi/h)x}{\sinh(\pi/h)a} \right)^{p+1} \left\{ \left( \int_{-\infty}^{\infty} |f(\zeta)|^{2} \, ds \right)^{1/2} + \left( \int_{-\infty}^{\infty} |f(\zeta)|^{2} \, ds \right)^{1/2} \right\}, \quad x \in \mathbb{R}.
\]

\[(1.13)\]
Proof. Let \( f \) be bounded by \( M \). Then we estimate the integrals

\[
\left| \int_{x}^{x+1(\pi/h)} f(\xi) \frac{d\xi}{\sin^{1+1}(\pi/h)\xi} \right| \leq \frac{2Ma}{(n + \frac{1}{2}h - |x|)}.
\]

Thus, by Lemma 1.2,

\[
f(x) = \lim_{n \to \infty} \sum_{m=-n}^{n} f^{(m)}(mh) \Delta_{e}(x - mh)
\]

(1.14) + \frac{1}{2\pi i} \sinh \frac{\pi}{h} x \left\{ \int_{-i - \pi/2 + \frac{1}{2}a}^{i + \pi/2 - \frac{1}{2}a} \frac{f(\xi) d\xi}{(\xi - x) \sin^{1+1}(\pi/h)\xi} - \int_{-i - \pi/2 - \frac{1}{2}a}^{i + \pi/2 + \frac{1}{2}a} \frac{f(\xi) d\xi}{(\xi - x) \sin^{1+1}(\pi/h)\xi} \right\}, \quad x \in \mathbb{R},
\]

where convergence is locally uniform for all \( x \in \mathbb{R} \). Upon noting that

\[
\left| \sin \frac{\pi}{h} \xi \right| \geq \sinh \frac{\pi}{h} a, \quad \xi = \xi \pm ia,
\]

and

\[
\int_{-\infty}^{\infty} ds/|\xi - x|^{2} = \pi/a,
\]

Schwarz's inequality yields

\[
\left| \int_{-\infty}^{\infty} \frac{f(\xi) d\xi}{(\xi - x) \sin^{1+1}(\pi/h)\xi} \right| \leq \frac{\sqrt{\pi}}{a \sinh^{1+1}(\pi/h)a} \left( \int_{-\infty}^{\infty} |f(\xi)|^{2} ds \right)^{1/2}, \quad x \in \mathbb{R}.
\]

Hence, letting \( n \to \infty \) in (1.14) completes the proof.

Remark. From the bound (1.13), we easily see that

\[
\lim_{h \to 0} T_{p,h}(f)(x) = f(x), \quad p = 0, 1, \ldots,
\]

and

\[
\lim_{h \to 0} T_{p,h}(f)(x) = f(x), \quad h > 0, \quad \sinh \frac{\pi}{h} a > 1,
\]

where convergence is uniform for all \( x \in \mathbb{R} \). In both cases \( h \to 0, \) \( p \) fixed and \( p \to \infty, \) \( h \) fixed, the convergence is exponential.

The following theorem describes a class of functions for which \( T_{p,h}(f) = f \) is true.

Theorem 1.3. Let \( f \) be an entire function, such that

(1.15) \[
|f(z)| \leq ce^{\rho |y|}, \quad z = x + iy \in \mathbb{C},
\]

with real numbers \( c \geq 0 \) and \( 0 \leq \rho < (p + 1)\pi/h \). Then the \( p \)th cardinal series for \( f \) with respect to \( h \) is locally uniformly convergent for all \( z \in \mathbb{C} \), and the identity

(1.16) \[
T_{p,h}(f)(z) = f(z), \quad z \in \mathbb{C},
\]

holds.

Proof. By (1.15) we have
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\[ \left| \frac{f(\xi)}{\sin^{\pi h}(\pi/h)\xi} \right| \leq \frac{c e^{\pi h}}{\sin^{\pi h+1}(\pi/h)\eta} \]

\[ = O\left(\exp\left[-\left(p + 1 \frac{\pi}{h} - \rho\right)\eta\right]\right), \quad \xi = \xi + i\eta \in \mathbb{C}, \]

as \( |\eta| \to \infty \), and therefore

\[ \lim_{\eta \to \infty} \int_{-(n+1/2)\pm i\eta}^{(n+1/2)\pm i\eta} \frac{f(\xi)}{(\xi - x)\sin^{\pi h+1}(\pi/h)\xi} \, d\xi = 0, \]

that is, by Lemma 1.2,

\[ f(\xi) = \sum_{m=-\infty}^{n} \sum_{q=0}^{p} f^{(q)}(mh)\eta(z - mh) \]

for all \( n \) with \( (n + \frac{1}{2})h > |x| \). Making use of

\[ \left| \sin \frac{\pi}{h} \xi \right| = \cosh \frac{\pi}{h} \eta \geq \frac{1}{2} \exp\left[\frac{\pi}{h}^2 \eta^2\right], \quad \xi = \pm(n + \frac{1}{2})h + i\eta, \]

we conclude that

\[ \int_{-(n+1/2)\pm i\eta}^{(n+1/2)\pm i\eta} \frac{f(\xi)}{(\xi - x)\sin^{\pi h+1}(\pi/h)\xi} \, d\xi \leq \frac{2^{p+2}c}{(p + 1)(\pi/h) - \rho(n + \frac{1}{2})h - |x|}. \]

Letting \( n \to \infty \) in (1.17), the assertion of the theorem follows.

Example. If we choose \( f(z) := e^{iz}, \ z \in \mathbb{C}, 0 \leq \rho < (p + 1)\pi/h \), we obtain the local uniform convergent expansion

\[ e^{iz} = \sum_{m=-\infty}^{n} \sum_{q=0}^{p} (ip)^q e^{imh} \eta(z - mh), \quad z \in \mathbb{C}. \]

Setting \( \rho := r\pi/h, r = 0, 1, \cdots, p \), we derive

\[ \exp\left[ir \frac{\pi}{h} z\right] = \sum_{m=-\infty}^{n} \sum_{q=0}^{p} \left(ir \frac{\pi}{h}\right)^q (-1)^m \eta(z - mh), \quad z \in \mathbb{C}. \]

2. Numerical Integration. We integrate the \( p \)th cardinal series of \( f \) termwise and obtain the series

\[ I_{p,\eta}(f) := h \sum_{m=-\infty}^{n} \sum_{q=0}^{p} \left(\frac{h}{2\pi}\right)^q a_{e,p} f^{(q)}(mh) \]

with the weights

\[ a_{e,p} := \frac{1}{2\pi} \left(\frac{2\pi}{h}\right)^{p+1} \int_{-\infty}^{\infty} t_e(x) \, dx, \quad q = 0, 2, \cdots, p. \]
If $q$ is odd, the integral (2.2) vanishes, since in this case the function $t_q$ is odd. The series (2.1) may be regarded as generalizations of the trapezoidal rule approximation for the integral $\int_a^b f(x) \, dx$.

In order to derive simple recurrence formulae for the weights $a_{q,p}$, we state

**THEOREM 2.1.** Let $p$ be even. Then the weights $a_{q,p}$ are uniquely determined by the identity

$$\sum_{q=0; q \text{ even}}^{p} a_{q,p} z^q = \prod_{q=1}^{p/2} (1 + (z/q)^q), \quad z \in \mathbb{C}. \tag{2.3}$$

**Proof.** Integrating (1.19), we have

$$\int_0^\infty \exp \left[ i r \frac{\pi}{h} x \right] dx = \sum_{q=0; q \text{ even}}^{p} \left( i r \frac{\pi}{h} \right)^q \int_{-\infty}^\infty t_q(x) dx, \quad r = 0, 2, \cdots, p,$$

thus, we are led to the system of $p/2 + 1$ linear equations

$$a_{0,p} = 1, \tag{2.4}$$

$$\sum_{q=0; q \text{ even}}^{p} (ir)^q a_{q,p} = 0, \quad r = 1, \cdots, p/2.$$

Since the determinant $D_p$ of (2.4) is a Vandermonde determinant with

$$D_p = i^{(p/2)(p/2+1)} \prod_{q>r=0}^{p/2} (q^2 - r^2) \neq 0,$$

the weights $a_{q,p}$ are uniquely determined by the system (2.4).

Define a polynomial $P_p$ of degree $p$ by

$$P_p(z) := \sum_{q=0; q \text{ even}}^{p} a_{q,p} z^q, \quad z \in \mathbb{C}.$$ 

Then (2.4) reads

$$P_p(0) = 1,$$

$$P_p(it) = 0, \quad r = \pm 1, \cdots, \pm p/2.$$

Hence $P_p(z) = \prod_{q=1}^{p/2} (1 + (z/q)^q)$, and (2.3) is established.

From (2.3) it follows that

$$(1 + (2z/p)^2) \sum_{q=0; q \text{ even}}^{p/2} a_{q,p-2} z^q = \sum_{q=0; q \text{ even}}^{p} a_{q,p} z^q, \quad p = 2, 4, \cdots.$$ 

Comparing the coefficients, we find the desired recursion formulae

$$a_{q,p} = 1, \quad p = 0, 2, \cdots,$$

$$a_{q,p} = a_{q,p-2} + (2/p)^q a_{q-2,p-2}, \quad q = 2, 4, \cdots, p - 2, \quad p = 2, 4, \cdots,$$ 

$$a_{p,p} = \frac{1}{((p/2)!)^2}, \quad p = 0, 2, \cdots.$$ 

Using (2.5), we obtain

$\star \Pi_{q=1}^{p/2}$ is to be interpreted as unity when $p = 0.$

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In the case of odd $p$, we integrate (1.19) for $r = 0, 2, \cdots, p - 1$ and get the system of $(p + 1)/2$ linear equations

$$a_{0,p} = 1,$$

(2.6)

$$\sum_{q=0, q \text{ even}}^{p-1} (ir)^{q} a_{q,p} = 0, \quad r = 1, \cdots, (p - 1)/2.$$

Comparing (2.6) with (2.4), we see that $a_{q,p} = a_{q,p-1}$, $q = 0, 2, \cdots, p - 1$. Thus, if $p$ is odd, $I_{p,k}(f) = I_{p-1,k}(f)$ is valid. Therefore, we may restrict ourselves to even $p$.

In the case when the function $f$ is analytic, we state a sufficient condition on the convergence of the series (2.1) and give a remainder in the following:

**Theorem 2.2.** Let the function $f$ be analytic in the strip $B := \mathbb{R} \times [-a, a] \subset \mathbb{C}$, $a > 0$, let $f(z) \to 0$, $z = x + iy$ as $x \to \pm \infty$ uniformly for all $-a \leq y \leq a$ and let

$$\int_{-\infty}^{\infty} |f(z)| \, ds < \infty, \quad \int_{-\infty}^{\infty} |f(z)| \, ds < \infty.$$

(2.7)

Then $f(x)$ exists, and the series (2.1) is convergent for even $p \geq 0$ and $h > 0$. The remainder

$$E_{p,k}(f) := \int_{-\infty}^{\infty} f(x) \, dx - I_{p,k}(f)$$

is given by

$$E_{p,k}(f) = \frac{1}{(2i)^{p+1}} \sum_{q=0}^{p/2} (-1)^{q} \left( \begin{array}{l} p + 1 \\ q \end{array} \right) \left( \int_{-\infty+ia}^{\infty+ia} \frac{\exp[i(p + 1 - 2q)(\pi/h)\xi]}{\sin^{p+1}(\pi/h)\xi} f(\xi) \, d\xi \right. $$

$$\left. - \int_{-\infty-ia}^{\infty-ia} \frac{\exp[-i(p + 1 - 2q)(\pi/h)\xi]}{\sin^{p+1}(\pi/h)\xi} f(\xi) \, d\xi \right\}$$

(2.9)

with the bound

$$|E_{p,k}(f)| \leq \frac{\exp[-(\pi/h)a]}{2 \sinh^{p+1}(\pi/h)\pi} \left( \int_{-\infty+ia}^{\infty+ia} |f(\xi)| \, ds + \int_{-\infty-ia}^{\infty-ia} |f(\xi)| \, ds \right).$$

(2.10)

**Proof.** From the assumption (2.7), we see by Cauchy’s theorem that $f(x)$ exists.

Using the identity

$$\sin^{p+1} x = \frac{\exp[(p + 1)(\pi/h)x]}{(2i)^{p+1}} \sum_{q=0}^{p/2} (-1)^{q} \left( \begin{array}{l} p + 1 \\ q \end{array} \right) \exp[-2iq(\pi/h)x],$$

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we deduce from the residue theorem that

\[ \frac{1}{\pi} \int_{-\infty}^{\infty} \sin^{p+1}(\pi/h) \frac{x - \xi}{x - \xi} \, dx = \exp[i(p + 1)(\pi/h)\xi] \sum_{q=0}^{p/2} (-1)^{(p + 1)/q} \exp[-2q(\pi/h)\xi]. \]

\[ \xi = \xi + ia, \]

\[ \frac{1}{\pi} \int_{-\infty}^{\infty} \sin^{p+1}(\pi/h) \frac{x - \xi}{x - \xi} \, dx = \exp[-i(p + 1)(\pi/h)\xi] \sum_{q=0}^{p/2} (-1)^{(p + 1)/q} \exp[2q(\pi/h)\xi]. \]

\[ \xi = \xi - ia. \]

From this the estimates

\[ (2.11) \quad \left| \frac{1}{\pi} \int_{-\infty}^{\infty} \sin^{p+1}(\pi/h) \frac{x - \xi}{x - \xi} \, dx \right| \leq \exp[-(p + 1)(\pi/h)a] \sum_{q=0}^{p/2} \left( \frac{p + 1}{q} \right) \exp[2q(\pi/h)a] \]

\[ \leq \exp[-(\pi/h)a], \quad \xi = \xi \pm ia, \]

follow.

Integrating (1.8) over \((-\infty, \infty)\) and interchanging the order of integration, we obtain

\[ \int_{-\infty}^{\infty} f(x) \, dx = \int_{-\infty}^{\infty} \left( \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \sin^{p+1}(\pi/h) \frac{x - \xi}{x - \xi} \, dx \right) d\xi. \]

With the aid of (2.11), we can estimate

\[ \left| \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(\xi)}{\sin^{p+1}(\pi/h)\xi} \left( \int_{-\infty}^{\infty} \sin^{p+1}(\pi/h) \frac{x - \xi}{x - \xi} \, dx \right) d\xi \right| \leq 2a \max_{\xi \in \{\pm(n + 1/2)h, i\eta\}} |f(\pm(n + 1/2)h + i\eta)|, \]

and

\[ \left| \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(\xi)}{\sin^{p+1}(\pi/h)\xi} \left( \int_{-\infty}^{\infty} \sin^{p+1}(\pi/h) \frac{x - \xi}{x - \xi} \, dx \right) d\xi \right| \leq \exp[-(\pi/h)a] \sin^{p+1}(\pi/h) \int_{-\infty}^{\infty} |f(\xi)| \, ds. \]

Thus, by the assumptions on \(f\), letting \(n \to \infty\) completes the proof.

**Remark.** From the bound (2.10), we have

\[ \lim_{h \to 0} I_{p,s}(f) = \int_{-\infty}^{\infty} f(x) \, dx, \quad p = 0, 2, \ldots, \]

and

\[ \lim_{p \to \infty} I_{p,s}(f) = \int_{-\infty}^{\infty} f(x) \, dx, \quad h > 0, \sinh \frac{\pi}{h} a > 1, \]

where convergence is exponential in both cases \(h \to 0, \text{fixed} \) and \(p \to \infty, h \text{fixed} \).