An Asymptotic Expansion of $W_{k,m}(z)$ with Large Variable and Parameters

By R. Wong*

Abstract. In this paper, we obtain an asymptotic expansion of the Whittaker function $W_{k,m}(z)$ when the parameters and variable are all large but subject to the growth restrictions that $k = o(z)$ and $m = o(z^{1/2})$ as $z \to \infty$. Here, it is assumed that $k$ and $m$ are real and $|\arg z| \leq \pi - \delta$.

1. Introduction. In this paper, we are concerned with the asymptotic behavior of the Whittaker function $W_{k,m}(z)$. This function depends on two parameters and a variable. When the parameters $k$ and $m$ are fixed and the variable $z$ is large, it is well known that a complete asymptotic expansion can be obtained; see [1, Section 7.1]. However, if the parameters $k$ and $m$ are allowed to increase without limit, the problem of finding asymptotic forms for $W_{k,m}(z)$ becomes much more involved and has been the subject of numerous investigations; see Buchholz [1], Chang, Chu and O'Brien [2], Kazarinoff [7], Erdélyi and Swanson [5], Slater [8] and the references given there. Although a great number of papers have been written on this subject, the treatment with two parameters and a variable is still incomplete.

In a recent paper [11], Wong and Rosenbloom have studied a certain inequality (see [4, p. 124]) connecting Whittaker functions and parabolic cylinder functions $D_0(z)$, and shown that this inequality can be improved considerably. However, the above-mentioned paper contains the restriction that $k$ and $m$ be again fixed. The purpose of this paper is to show that this condition can be relaxed so that $k$ and $m$ may depend on $z$. Moreover, we give a complete asymptotic expansion of $W_{k,m}(z)$ when the parameters and the variable are all large, i.e.,

\begin{equation}
(1.1) \quad k, m \quad \text{and} \quad z \to \infty
\end{equation}

but subject to the growth restrictions that

\begin{equation}
(1.2) \quad k = o(z) \quad \text{and} \quad m = o(z^{1/2}) \quad \text{as} \quad z \to \infty.
\end{equation}

Here, it is supposed that $k$ and $m$ are real and $|\arg z| \leq \pi - \delta$. The term "asymptotic" is used in the sense of Erdélyi and Wyman [6], which is more general than the usual Poincaré sense. This distinction is made clear in the theorems.

Received July 21, 1972.


Key words and phrases. Whittaker function, asymptotic expansion, parabolic cylinder functions, Hankel functions.

* Research partially supported by the National Research Council of Canada under contract No. A7359.

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2. Two Auxiliary Results. It is well known that Hankel functions $H^{(1)}_\nu(z)$ and $H^{(2)}_\nu(z)$ have the asymptotic expansions

\begin{equation}
H^{(1)}_\nu(z) = \left( \frac{2}{\pi z} \right)^{1/2} e^{i\left(z - \nu\pi/2 - \nu\pi/4\right)} \left\{ \sum_{m=0}^{\nu - 1} \frac{(-1)^{m}}{m!} \frac{(\nu, m)}{(2i)^m} + R^{(1)}_\nu \right\}
\end{equation}

and

\begin{equation}
H^{(2)}_\nu(z) = \left( \frac{2}{\pi z} \right)^{1/2} e^{-i\left(z - \nu\pi/2 - \nu\pi/4\right)} \left\{ \sum_{m=0}^{\nu - 1} \frac{(\nu, m)}{(2i)^m} + R^{(2)}_\nu \right\},
\end{equation}

where

\begin{equation}
(\nu, m) = \frac{\left\{ 4\nu^2 - 1 \right\} \left\{ 4\nu^2 - 3 \right\} \cdots \left\{ 4\nu^2 - (2m - 1)^2 \right\}}{2^{2m} m!},
\end{equation}

\begin{equation}
(\nu, 0) = 1,
\end{equation}

and the remainders $R^{(1)}_\nu$ and $R^{(2)}_\nu$ are both $O(z^{-\nu})$ when $\nu$ is a fixed number. For the results to be obtained, the following estimate is needed.

**Lemma 1.** Let $\arg z$ be restricted to the interval $[-\pi/2, 3\pi/2]$, and $\nu$ be a real-valued function of $z$ satisfying $\nu = o(z^{1/2})$ as $z \to \infty$. Then, for $i = 1$ and 2,

\begin{equation}
R^{(i)}_\nu = O(|\nu, p|/z^p), \quad as \ z \to \infty.
\end{equation}

**Proof.** We suppose first that $\nu \geq 0$ and $\text{Re } z \geq 0$. Under these conditions, Weber [9, Section 7.33] showed that

\begin{equation}
|R^{(i)}_\nu| \leq 2G^2 |(\nu, p)| \left\{ \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}p + \frac{1}{2})}{\Gamma(\frac{1}{2}p + \frac{1}{2})} \right\} |2z|^p
\end{equation}

where

\begin{equation}G = \left( 1 - \nu - \frac{1}{2} \right)^{-\nu - 1/2} (\nu > \frac{1}{2}),\end{equation}

\begin{equation}G = \left( 1 - \nu + \frac{3}{2} \right)^{-\nu - 5/2} \left( 1 + \frac{2\nu + 2}{r} \right) (\nu \leq \frac{1}{2}),\end{equation}

and $|z| = r$.

Since $G$ is clearly bounded when $0 \leq \nu \leq 1$ and $r$ is sufficiently large, we may assume that $1 < \nu \leq r^{1/2}$. A simple estimate then gives

\begin{equation}(-\nu - \frac{1}{2}) \log(1 - 1/2r^{1/2}) \leq (\nu + \frac{1}{2})/r^{1/2} \leq \frac{3}{2},\end{equation}

from which it follows that

\begin{equation}G \leq (1 - 1/2r^{1/2})^{-\nu - 1/2} \leq e^{3/2}.
\end{equation}

Therefore, a constant $A_\nu$ exists, which is independent of $\nu$ and $z$, such that

\begin{equation}|R^{(i)}_\nu| \leq A_\nu |(\nu, p)|/|z|^p \quad (i = 1, 2),\end{equation}

for all sufficiently large values of $z$. This is equivalent to (2.5).

Since $(\nu, p)$ is an even function of $\nu$, it follows from the identities [9, Section 3.61]

\begin{equation}H^{(1)}_{\nu}(z) = e^{x \nu i} H^{(1)}_\nu(\nu, z), \quad H^{(1)}_{\nu}(z) = e^{-x \nu i} H^{(1)}_{\nu}(z),
\end{equation}
and \([9, \text{Section 3.62}]\)

\[
H^{(1)}(\nu z^i) = -e^{-\nu i} H^{(2)}(\nu z),
\]

that the restrictions \(\nu \geq 0\) and \(\Re z \geq 0\) are unnecessary. Therefore, inequality (2.10) holds for all real values of \(\nu\) and complex \(z\) restricted to the sector \(-\pi/2 \leq \arg z \leq 3\pi/2\), as long as \(\nu = o(z^{1/2})\) as \(z \to \infty\). This completes the proof of Lemma 1.

Remark. It should be observed that no hypothesis has been made in the estimates concerning the relative values of \(\nu\) and \(p\); in this respect, Weber's result differs from that of Schläfli \([9, \text{Section 7.4}]\) which was used in our previous paper \([11]\).

In \([6]\), Erdélyi and Wyman have given an elegant proof of a result from which it is easily deduced that the parabolic cylinder function \(D_{-\lambda}(z)\) has the generalized asymptotic expansion

\[
z^\lambda e^{z^2/4} D_{-\lambda}(z) \sim \sum_{n=0}^{\infty} \frac{(-1)^n(\lambda z)^{2n}}{n!(2z)^n} \cdot \left\{ \frac{\lambda^{2n}}{z^n} \right\},
\]

as \(z \to \infty\) in \(|\arg z| \leq \pi/2 - \Delta\), where \(\lambda > 0\) and \(\lambda = o(z)\). The meaning of (2.11) is

\[
z^\lambda e^{z^2/4} D_{-\lambda}(z) = \sum_{n=0}^{N} \frac{(-1)^n(\lambda z)^{2n}}{n!(2z)^n} + o\left( \frac{\lambda^{2N}}{z^n} \right)
\]

as \(z \to \infty\), for every fixed integer \(N \geq 0\), where the \(o\)-symbol is independent of \(\lambda\) and \(z\). Unfortunately, they proved the result only for \(\lambda > 0\), while, for our results, we want to use all real values of \(\lambda\). Although the conditions \(\lambda > 0\) and \(|\arg z| \leq \pi/2 - \Delta\) in (2.13) can be easily weakened to \(|\arg \lambda| \leq \pi/2 - \Delta\) and \(|\arg z| \leq 3\pi/2 - \Delta\), their proof does not seem readily adapted to extensions allowing \(\lambda\) to be negative.

The following lemma shows that the condition \(\lambda > 0\) is indeed unnecessary.

Lemma 2. The result in (2.13) is true if "\(\lambda > 0\)" is replaced by "\(\lambda\) real".

Proof. We start with the contour integral representation

\[
e^{z^2/4} D_{-\lambda}(z) = -\frac{\Gamma(1 - \lambda)}{2\pi i} \int_{C} (-t)^{\lambda-1} e^{-t^2/z^2} dt,
\]

where the path of integration starts at \(+\infty\), goes around the origin once in the positive direction and returns to \(+\infty\). The integrand is rendered one-valued by taking \(-\pi \leq \arg (-t) \leq \pi\).

Since it has already been shown that (2.13) holds when \(\lambda\) is finite or \(\lambda > 0\) but \(\lambda = o(z)\), we shall assume that \(\lambda\) is large and negative. Let \(r_n(t)\), \(N = 0, 1, 2, \cdots\), be defined by the relation

\[
e^{-t^2/2} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{2^n \cdot n!} + r_n(t).
\]

It is evident that, if \(t\) is restricted to the path of integration, a constant \(B_N\) can be found such that

\[
|r_n(t)| \leq B_N |t|^{2N+2}.
\]

Substituting (2.16) in (2.15) and integrating term by term, we obtain

\[
e^{z^2/4} D_{-\lambda}(z) = \sum_{n=0}^{N} \frac{(-1)^n(\lambda z)^{2n}}{2^n \cdot n!} z^{-\lambda-2n} + \Gamma(1 - \lambda)e_\nu(\lambda, z),
\]
where
\[
|\epsilon_n(\lambda, z)| \leq \frac{1}{2\pi} \int_{-\infty}^{(0+)} |(-t)^{\lambda-1} r_n(t) e^{-zt} \, dt|
\]
(2.19)
\[
\leq \frac{B_n}{2\pi} \int_{-\infty}^{(0+)} |t^{\lambda+2N+1} e^{-zt} \, dt|
\]
by (2.17). Since \(\lambda\) is negative, the transformation \(zt = (-\lambda)\tau\) gives
\[
\int_{-\infty}^{(0+)} |t^{\lambda+2N+1} e^{-zt} \, dt| = \left|\frac{\lambda}{z}\right| \int_{-\infty}^{(0+)} |t^{\lambda+2N+1} e^{\lambda r} \, d\tau|
\]
when \(z\) is real and positive. It is not difficult to see that (2.20) in fact holds when \(|\arg z| < \pi/2\). Hence,
\[
\left|\frac{\lambda}{z}\right| |\epsilon_n(\lambda, z)| \leq \frac{B_n}{2\pi} \int_{-\infty}^{(0+)} |t^{2N+1} e^{\lambda(r+1\log r)} \, d\tau|
\]
valid when \(\lambda < 0\) and \(|\arg z| \leq \pi/2 - \Delta\). To the last integral, we apply the method of steepest descents [3, Section 30]. Hence,
\[
\int_{-\infty}^{(0+)} |t^{2N+1} e^{\lambda(r+1\log r)} \, d\tau| \sim e^{-\lambda [-\pi/2\lambda]^{1/2}},
\]
as \(\lambda \to -\infty\). Coupling the results (2.21) and (2.22), we obtain
\[
z^\lambda \epsilon_n(\lambda, z) = O\left\{\left((-\lambda/z)^{2N+2} e^{-\lambda(-\lambda)^{\lambda-1/2}}\right)\right\},
\]
as \(z \to \infty\) in \(|\arg z| \leq \pi/2 - \delta\), where the \(O\)-symbol is independent of \(\lambda\) and \(z\). Finally, by Stirling's formula
\[
\Gamma(1 - \lambda) z^\lambda \epsilon_n(\lambda, z) = O\left\{\left(\lambda/z\right)^{2N+2}\right\}
\]
and so the lemma is established.

**Remark.** The above analysis can be used to give similar expansions for the derivatives of \(D_f z\) with respect to \(z\). In particular, we have
\[
D_f z(\lambda) \sim \left(-\frac{1}{3}\right) z^{1-\lambda} e^{-z/4}, \quad \text{as } z \to \infty \text{ in } |\arg z| \leq \pi/2 - \delta,
\]
where \(\lambda\) is real and \(\lambda = o(z)\).

3. **Main Theorem.** It is known that the Whittaker function has the integral representation [1, Section 5.3]
\[
W_{k,m}(z^2) = z e^{z^2/2+ (m+1/2-k) \times i} \int_{-\infty}^{\infty} e^{-u^2} H_{2m}^{(1)}(2zu) u^{2k} \, du,
\]
(3.1)
where the path of integration runs from \(-\infty\) to \(\infty\) and passes above the singularity at the origin. If we substitute (2.1) for \(H_{2m}^{(1)}\), we obtain
\[
W_{k,m}(z^2) = 2^{1/4-k} \sqrt{z} \left\{ \sum_{r=0}^{k-1} \frac{(2m, r)}{(2z \sqrt{2})} D_{2k-r-1/2}(z \sqrt{2}) + E_k(z) \right\}
\]
(3.2)
where the remainder is given by
(3.3) \[ E_p(z) = \frac{1}{\sqrt{\pi}} 2^{k-1/4} e^{(1/4-k) \pi i + \pi/2} \int_{-\infty}^{\infty} e^{-u^4 + 2i u z \eta_0} u^{2k-1/2} R_p^{(1)}(2zu) \, du. \]

This result is well known [4, p. 124]. When \( k \) and \( m \) are fixed, it was shown in [11, (3.1)] that \( E_p(z) = O(e^{-z^{1/2} z^{2k-2p-1/2}}) \), uniformly in \( \arg z \), as \( z \to \infty \) in \( |\arg z| \leq \pi/4 - \Delta \).

When \( k \) and \( m \) are functions of \( z \), we have the following lemma.

**Lemma 3.** Let \( k \) and \( m \) be real-valued functions of \( z \) for which \( k = o(z) \) and \( m = o(z^{1/2}) \) as \( |z| \to \infty \). If \( |m| \geq \delta > 0 \) then

(3.4) \[ E_p(z) = O\left(2^{k-1/2} e^{-z^{1/2}} (m/z)^{2p}\right). \]

If \( |m| \leq \delta \) then

(3.5) \[ E_p(z) = O\left(2^{k-1/2} e^{-z^{1/2}} (m/z)^{2p-1/2}\right). \]

Both results hold uniformly in \( \arg z \), as \( z \to \infty \) in \( |\arg z| \leq \pi/2 - \Delta \), and the constants implied in \( O \)-symbols are independent of \( k \), \( m \), and \( z \).

**Proof.** Returning to (3.3), we let

(3.6) \[ I = \int_{-\infty}^{\infty} e^{-u^4 + 2i u z \eta_0} u^{2k-1/2} R_p^{(1)}(2zu) \, du. \]

In [11], it was shown that by a change of variable \( u = zu' \) followed by a deformation of the contour,

(3.7) \[ I = z^{2k+1/2} \int_{-\infty}^{\infty} e^{-z^4(x^4+1)}(x + i)^{2k-1/2} R_p^{(1)}(2z^2(x + i)) \, dx, \]

the path of integration now being a straight line joining \(-\infty \) to \( \infty \). By Lemma 1,

(3.8) \[ |I| \leq A_p |(2m, p)| \left| e^{-z^4} z^{2k-2p+1/2} \right| J, \]

where

(3.9) \[ J = \int_{-\infty}^{\infty} |e^{-z^4}(x + i)^{2k-1/2} \, dx| \]

and the constant \( A_p \) depends only on \( p \). Since \( x \) is real, we have \( |x + i| \geq 1 \), and so

(3.10) \[ J \leq 2 \int_{0}^{\infty} e^{-(\Re x^4)} (x^2 + 1)^k \, dx. \]

We consider separately the cases \( k \leq 0 \) and \( k > 0 \).

When \( k \leq 0 \),

(3.11) \[ J \leq 2 \int_{0}^{\infty} e^{-(\Re x^4)x^4} \, dx = \left(\frac{\pi}{\Re x^2}\right)^{1/2}. \]

Hence, \( J = O(z^{-1}) \) for \( z \) restricted to \( |\arg z| \leq \pi/4 - \Delta \).

When \( k > 0 \),

(3.12) \[ J \leq 2 \int_{0}^{\infty} e^{-(\Re x^4-k)x^4} \, dx \]

provided that the integral exists. Since \( k = o(z) \) as \( |z| \to \infty \),
for sufficiently large \( z \) in the sector \( |\arg z| \leq \pi/4 - \Delta \), where \( \eta_k \) is a positive finite number and independent of \( |z| \). Therefore, we again have \( J = O(z^{-1}) \), as \( z \to \infty \) in \( |\arg z| \leq \pi/4 - \Delta \).

We have thus proved that a constant \( A'_e \) exists such that

\[
|I| \leq A'_e |(2m, p)e^{-z^{2k-2p-1/2}}|, \tag{3.14}
\]

for large values of \( z \) in \( |\arg z| \leq \pi/4 - \Delta \). The region of validity can be extended to \( |\arg z| \leq \pi/2 - \Delta \) by a standard argument. We rotate the path of integration in (3.7) through an arbitrary angle \( \gamma \), where \( -\pi/4 < \gamma < \pi/4 \). When \( z \) is positive, use of Cauchy’s theorem easily shows that (3.7) is valid if the upper and lower limits are replaced by \( \infty e^{i\gamma} \) and \( -\infty e^{i\gamma} \) respectively. With this change, (3.7) holds when \( |\arg (ze^{i\gamma})| \leq \pi/4 - \Delta \). A repetition of the proof (with some slight modifications) then shows that (3.14) is also valid in this angle. By varying \( \gamma \), it follows that (3.14) holds when \( |\arg z| \leq \pi/2 - \Delta \).

Since \( E_\delta(z) = (1/\sqrt{\pi})2^{k-1/4}e^{-(1/4-k)z^{2k-2p-1/2}I} \), by (3.14),

\[
E_\delta(z) = O\left(2^{k} (2m, p)e^{-z^{2k-2p-1/2}}\right) \tag{3.15}
\]

for all large values of \( z \) restricted to the sector \( |\arg z| \leq \pi/2 - \Delta \). When \( |m| \leq \delta \), (3.15) is certainly equivalent to (3.5). When \( |m| \geq \delta > 0 \), (3.4) follows from (3.15) in view of the fact that \( (2m, p) \sim (2m)^{2k}/p! \).

**Main Theorem.** Let \( k \) and \( m \) be real-valued functions of \( z \) satisfying conditions (1.1) and (1.2). Then, for any \( N \geq 0 \),

\[
2^{k-1/4} W_{k, m}(z) = D_{2k-1/2}((2z)^{1/2}) \left[ \sum_{s=0}^{N+1} \frac{a_s}{z^s} + o\left(\left(\frac{m^2}{z}\right)^{2N+2}\right)\right] + D_{2k-1/2}((2z)^{1/2}) \left[ \sum_{s=0}^{N} \frac{b_s}{z^s} + o\left(\left(\frac{m^2}{z}\right)^{2N+2}\right)\right] \tag{3.16}
\]

as \( z \to \infty \) in \( |\arg z| \leq \pi - \delta \), uniformly with respect to \( \arg z \). The coefficients \( a_s \) and \( b_s \) depend on \( k \) and \( m \), and are explicitly given in (3.24).

**Proof.** Clearly, \( |(m^2/z)^{2s}| \) is an asymptotic sequence under the hypothesis \( m = o(z^{1/2}) \) as \( z \to \infty \). Let \( N \) be an arbitrary but fixed positive integer, and set

\[
S = \sum_{s=0}^{2N+2} \frac{(2m, r)}{(2(2s)^{1/2})} D_{2k-s-1/2}((2z)^{1/2}). \tag{3.17}
\]

The following lemma is given in [10].

**Lemma.** For each \( r \geq 0 \) we have

\[
(-1)^r (-\lambda), D_{\lambda-r}(z) = D_{\lambda}(z) P_r(z) + D_{\lambda}(z) Q_{r-1}(z), \tag{3.18}
\]

where \( P_r(z) \) and \( Q_{r-1}(z) \) are polynomials of the form

\[
P_r(z) = \sum_{s=0}^{\lfloor r/2 \rfloor} p_r, z^{r-2s}, \tag{3.19}
\]

\[
Q_{r-1}(z) = \sum_{s=0}^{\lfloor (r-1)/2 \rfloor} q_{r-1}, z^{r-(2s+1)} \tag{3.20}
\]
The coefficients \( p_{r,s} \) and \( q_{r-1,s} \) can be successively determined from the recurrence relations

\begin{align}
\tag{3.21}
P_{r+1}(z) &= zP_r(z) + (-\lambda + r - 1)P_{r-1}(z), \\
\tag{3.22}
Q_{r}(z) &= zQ_{r-1}(z) + (-\lambda + r - 1)Q_{r-2}(z),
\end{align}

with \( P_0(z) = 1, \ P_1(z) = z/2, \ Q_{-1}(z) = 0 \) and \( Q_0(z) = 1 \).

Now, let \( |k| \geq N + 1 \) so that \( 2k - \frac{1}{2} \neq 0, 1, \ldots, 2N + 1, \) and hence \( (\frac{3}{2} - 2k) \neq 0 \) for \( r = 0, 1, \ldots, 2N + 2 \). It follows from (3.17) that the sum \( S \) can be rearranged in the form

\begin{align}
\tag{3.23}
S &= D_{2k-1/2}((2z)^{1/2}) \sum_{s=0}^{N+1} \frac{a_s}{z^s} + D'_{2k-1/2}((2z)^{1/2}) \sum_{t=0}^{N} \frac{b_t}{z^{t+1/2}}
\end{align}

where

\begin{align}
\tag{3.24}
a_s &= \frac{1}{2^t} \sum_{r \geq 2k} \frac{(-1)^t(2m, r)}{2^t(\frac{1}{2} - 2k)} p_{r,s}, \quad \text{and} \\
b_s &= \frac{1}{2^{s+1/2}} \sum_{r \geq 2k+1} \frac{(-1)^t(2m, r)}{2^t(\frac{1}{2} - 2k)} q_{r-1,s}.
\end{align}

Therefore

\begin{align}
\tag{3.25}
W_{k,m}(z) &= 2^{1/4-k}z^{1/4} \left\{ D_{2k-1/2}((2z)^{1/2}) \sum_{s=0}^{N+1} \frac{a_s}{z^s} \\
&\quad + D'_{2k-1/2}((2z)^{1/2}) \sum_{t=0}^{N} \frac{b_t}{z^{t+1/2}} + E_{2N+3}(\sqrt{z}) \right\}
\end{align}

for any fixed integer \( N \geq 0 \).

Now, it only remains to consider the remainder \( E_{2N+3} \). By Lemmas 2 and 3, we have

\begin{align}
\tag{3.26}
E_{2N+3}(\sqrt{z}) &= O\left( (m^2/z)^{2N+3} D_{2k-1/2}((2z)^{1/2}) \right),
\end{align}

and, similarly,

\begin{align}
\tag{3.27}
E_{2N+3}(\sqrt{z}) &= O\left( (m^2/z)^{2N+3} z^{-1/2} D'_{2k-1/2}((2z)^{1/2}) \right)
\end{align}

by (3.26). Both results hold uniformly with respect to \( \arg z \), as \( z \to \infty \) in \( |\arg z| \leq \pi - \delta \).

We have thus proved that, for any integer \( N \geq 0 \),

\begin{align}
\tag{3.28}
2^{k-1/4} W_{k,m}(z) &= \frac{D_{2k-1/2}((2z)^{1/2})}{z^{-1/4}} \left[ \sum_{s=0}^{N+1} \frac{a_s}{z^s} + O\left( (m^2/z)^{2N+3} \right) \right] \\
&\quad + D'_{2k-1/2}((2z)^{1/2}) \left[ \sum_{t=0}^{N} \frac{b_t}{z^{t+1/2}} + O\left( (m^2/z)^{2N+3} \right) \right],
\end{align}

as \( z \to \infty \) in \( |\arg z| \leq \pi - \delta \), uniformly with respect to \( \arg z \), which certainly implies the required result.

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