An Asymptotic Expansion of $W_{k,m}(z)$ with Large Variable and Parameters

By R. Wong*

Abstract. In this paper, we obtain an asymptotic expansion of the Whittaker function $W_{k,m}(z)$ when the parameters and variable are all large but subject to the growth restrictions that $k = o(z)$ and $m = o(z^{1/2})$ as $z \to \infty$. Here, it is assumed that $k$ and $m$ are real and $|\arg z| \leq \pi - \delta$.

1. Introduction. In this paper, we are concerned with the asymptotic behavior of the Whittaker function $W_{k,m}(z)$. This function depends on two parameters and a variable. When the parameters $k$ and $m$ are fixed and the variable $z$ is large, it is well known that a complete asymptotic expansion can be obtained; see [1, Section 7.1]. However, if the parameters $k$ and $m$ are allowed to increase without limit, the problem of finding asymptotic forms for $W_{k,m}(z)$ becomes much more involved and has been the subject of numerous investigations; see Buchholz [1], Chang, Chu and O'Brien [2], Kazarinoff [7], Erdélyi and Swanson [5], Slater [8] and the references given there. Although a great number of papers have been written on this subject, the treatment with two parameters and a variable is still incomplete.

In a recent paper [11], Wong and Rosenbloom have studied a certain inequality (see [4, p. 124]) connecting Whittaker functions and parabolic cylinder functions $D_0(z)$, and shown that this inequality can be improved considerably. However, the above-mentioned paper contains the restriction that $k$ and $m$ be again fixed. The purpose of this paper is to show that this condition can be relaxed so that $k$ and $m$ may depend on $z$. Moreover, we give a complete asymptotic expansion of $W_{k,m}(z)$ when the parameters and the variable are all large, i.e.,

\begin{equation}
(1.1)
\begin{array}{c}
k, m \quad \text{and} \quad z \to \infty
\end{array}
\end{equation}

but subject to the growth restrictions that

\begin{equation}
(1.2)
\begin{array}{c}
k = o(z) \quad \text{and} \quad m = o(z^{1/2}) \quad \text{as} \quad z \to \infty.
\end{array}
\end{equation}

Here, it is supposed that $k$ and $m$ are real and $|\arg z| \leq \pi - \delta$. The term "asymptotic" is used in the sense of Erdélyi and Wyman [6], which is more general than the usual Poincaré sense. This distinction is made clear in the theorems.

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2. Two Auxiliary Results. It is well known that Hankel functions \( H^{(1)}(z) \) and \( H^{(2)}(z) \) have the asymptotic expansions

\[
H^{(1)}(z) = \left( \frac{2}{\pi z} \right)^{1/2} e^{i(z - \pi/2 - \pi/4)} \left\{ \sum_{m=0}^{p-1} \frac{(-1)^m (\nu, m)}{(2iz)^m} + R_p^{(1)} \right\}
\]

and

\[
H^{(2)}(z) = \left( \frac{2}{\pi z} \right)^{1/2} e^{-i(z - \pi/2 - \pi/4)} \left\{ \sum_{m=0}^{p-1} \frac{(\nu, m)}{(2iz)^m} + R_p^{(2)} \right\},
\]

where

\[
(\nu, m) = \frac{\{4\nu^2 - 1\} \{4\nu^2 - 3\} \cdots \{4\nu^2 - (2m - 1)^2\}}{2^{2m} m!},
\]

\[
(\nu, 0) = 1,
\]

and the remainders \( R_p^{(1)} \) and \( R_p^{(2)} \) are both \( O(z^{-p}) \) when \( \nu \) is a fixed number. For the results to be obtained, the following estimate is needed.

\textbf{Lemma 1.} Let arg \( z \) be restricted to the interval \([-\pi/2, 3\pi/2]\], and \( \nu \) be a real-valued function of \( z \) satisfying \( \nu = o(z^{1/2}) \) as \( z \to \infty \). Then, for \( i = 1 \) and \( 2 \),

\[
R_p^{(i)} = O((\nu, p)/z^p), \quad \text{as} \quad z \to \infty.
\]

\textbf{Proof.} We suppose first that \( \nu \geq 0 \) and Re \( z \geq 0 \). Under these conditions, Weber [9, Section 7.33] showed that

\[
|R_p^{(i)}| \leq 2G^2 |(\nu, p)| \frac{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{3}{2}p + 1\right)}{\Gamma\left(\frac{1}{2} + p + \frac{3}{2}\right) |2z|^p} \quad (i = 1, 2),
\]

where

\[
G = \left( 1 - \frac{\nu + \frac{3}{2}}{2r} \right)^{-r/2} \quad (\nu > \frac{1}{2}),
\]

\[
G = \left( 1 - \frac{\nu - \frac{1}{2}}{2r} \right)^{-r/2} \left( 1 + \frac{2\nu + 2}{r} \right) \quad (\nu \leq \frac{1}{2}),
\]

and \( |z| = r \).

Since \( G \) is clearly bounded when \( 0 \leq \nu \leq 1 \) and \( r \) is sufficiently large, we may assume that \( 1 < \nu \leq r^{1/2} \). A simple estimate then gives

\[
(-\nu - \frac{1}{2}) \log(1 - 1/2r^{1/2}) \leq (\nu + \frac{1}{2})/r^{1/2} \leq \frac{3}{2}
\]

from which it follows that

\[
G \leq (1 - 1/2r^{1/2})^{-r/2} \leq e^{3/2}.
\]

Therefore, a constant \( A_\nu \) exists, which is independent of \( \nu \) and \( z \), such that

\[
|R_p^{(i)}| \leq A_\nu |(\nu, p)|/|z|^p \quad (i = 1, 2),
\]

for all sufficiently large values of \( z \). This is equivalent to (2.5).

Since \( (\nu, p) \) is an even function of \( \nu \), it follows from the identities [9, Section 3.61]

\[
H^{(1)}(z) = e^{x\nu i} H^{(1)}(z), \quad H^{(2)}(z) = e^{-x\nu i} H^{(2)}(z)
\]
and [9, Section 3.62]

\( H^{(1)}(ze^{-i}) = -e^{-\nu i} H^{(2)}(z) \),

that the restrictions \( \nu \geq 0 \) and \( \Re z \geq 0 \) are unnecessary. Therefore, inequality (2.10) holds for all real values of \( \nu \) and complex \( z \) restricted to the sector \( -\pi/2 \leq \arg z \leq 3\pi/2 \), as long as \( \nu = o(z^{1/2}) \) as \( z \to \infty \). This completes the proof of Lemma 1.

**Remark.** It should be observed that no hypothesis has been made in the estimates concerning the relative values of \( \nu \) and \( \rho \); in this respect, Weber’s result differs from that of Schläfli [9, Section 7.4] which was used in our previous paper [11].

In [6], Erdélyi and Wyman have given an elegant proof of a result from which it is easily deduced that the parabolic cylinder function \( D_{-\lambda}(z) \) has the generalized asymptotic expansion

\[
\sim \sum_{n=0}^{\infty} \frac{(-1)^n(\lambda)_{2n}}{n!(2z)^n} \cdot \left( \frac{\lambda}{z} \right)^{2n},
\]

as \( z \to \infty \) in \( |\arg z| \leq \pi/2 - \Delta \), where \( \lambda > 0 \) and \( \lambda = o(z) \). The meaning of (2.11) is

\[
z^{\nu/4} D_{-\lambda}(z) = \sum_{n=0}^{N} \frac{(-1)^n(\lambda)_{2n}}{n!(2z)^n} + o\left( \frac{\lambda}{z^{2N}} \right)
\]

as \( z \to \infty \), for every fixed integer \( N \geq 0 \), where the \( o \)-symbol is independent of \( \lambda \) and \( z \). Unfortunately, they proved the result only for \( \lambda > 0 \), while, for our results, we want to use all real values of \( \lambda \). Although the conditions \( \lambda > 0 \) and \( |\arg z| \leq \pi/2 - \Delta \) in (2.13) can be easily weakened to \( |\arg \lambda| \leq \pi/2 - \Delta \) and \( |\arg z| \leq 3\pi/2 - \Delta \), their proof does not seem readily adapted to extensions allowing \( \lambda \) to be negative. The following lemma shows that the condition \( \lambda > 0 \) is indeed unnecessary.

**Lemma 2.** The result in (2.13) is true if "\( \lambda > 0 \)" is replaced by "\( \lambda \) real".

**Proof.** We start with the contour integral representation

\[
e^{\nu/4} D_{-\lambda}(z) = -\frac{\Gamma(1 - \lambda)}{2\pi i} \int_{-\infty}^{(0+)} (-t)^{-\nu/2 - 1} e^{-t/2 - zt} dt,
\]

where the path of integration starts at \(+\infty\), goes around the origin once in the positive direction and returns to \(+\infty\). The integrand is rendered one-valued by taking \(-\pi \leq \arg (-t) \leq \pi\).

Since it has already been shown that (2.13) holds when \( \lambda \) is finite or \( \lambda > 0 \) but \( \lambda = o(z) \), we shall assume that \( \lambda \) is large and negative. Let \( r_N(t) \), \( N = 0, 1, 2, \ldots \), be defined by the relation

\[
e^{-t/2} = \sum_{n=0}^{N} \frac{(-1)^n t^{2n}}{2^n n!} + r_N(t).
\]

It is evident that, if \( t \) is restricted to the path of integration, a constant \( B_N \) can be found such that

\[
|r_N(t)| \leq B_N |t|^{2N+2}.
\]

Substituting (2.16) in (2.15) and integrating term by term, we obtain

\[
e^{t/4} D_{-\lambda}(z) = \sum_{n=0}^{N} \frac{(-1)^n(\lambda)_{2n}}{2^n n!} z^{-\lambda - 2n} + \Gamma(1 - \lambda)\epsilon_N(\lambda, z),
\]
where

\[
|e_N(\lambda, z)| \leq \frac{1}{2\pi} \int_{-\infty}^{(0+)} \left|(-t)^{\lambda-1} r_N(t)e^{-zt} \right| dt
\]

by (2.17). Since \( \lambda \) is negative, the transformation \( zt = (-\lambda)\tau \) gives

\[
\int_{-\infty}^{(0+)} \left|t^{\lambda+2N+1} e^{-zt} \right| dt = \frac{B_N}{2\pi} \int_{-\infty}^{(0+)} \left|\tau^{\lambda+2N+1} e^{\lambda\tau} \right| d\tau
\]

when \( z \) is real and positive. It is not difficult to see that (2.20) in fact holds when \(|\arg z| < \pi/2\). Hence,

\[
\int_{-\infty}^{(0+)} \left|t^{\lambda+2N+1} e^{-zt} \right| dt = \frac{B_N}{2\pi} \int_{-\infty}^{(0+)} \left|\tau^{\lambda+2N+1} e^{\lambda\tau} \right| d\tau
\]

valid when \( \lambda < 0 \) and \(|\arg z| \leq \pi/2 - \Delta\). To the last integral, we apply the method of steepest descents [3, Section 30]. Hence,

\[
\int_{-\infty}^{(0+)} \left|\tau^{2N+1} e^{\lambda(\tau + \log \tau)} \right| d\tau \sim e^{\lambda\left[-\pi/2\lambda\right]^{1/2}},
\]

as \( \lambda \to -\infty \). Coupling the results (2.21) and (2.22), we obtain

\[
z^{\lambda} e_N(\lambda, z) = O\left(\frac{1}{z}\right)^{2N+2} e^{-\lambda(\lambda-1)/2},
\]

as \( z \to \infty \) in \(|\arg z| \leq \pi/2 - \delta\), where the \( O \)-symbol is independent of \( \lambda \) and \( z \). Finally, by Stirling's formula

\[
\Gamma(1-\lambda)z^{\lambda} e_N(\lambda, z) = O\left(\frac{1}{z}^{2N+2}\right)
\]

and so the lemma is established.

**Remark.** The above analysis can be used to give similar expansions for the derivatives of \( D_\lambda(z) \) with respect to \( z \). In particular, we have

\[
D_\lambda(z) \sim (-\frac{1}{2})z^{-\lambda-1} e^{-z^2/4}, \quad \text{as } z \to \infty \text{ in } |\arg z| \leq \pi/2 - \delta,
\]

where \( \lambda \) is real and \( \lambda = o(z) \).

3. **Main Theorem.** It is known that the Whittaker function has the integral representation [1, Section 5.3]

\[
W_{k,m}(z^2) = ze^{z^2/2 + (m+1/2-k)^2} i \int_{-\infty}^{\infty} e^{-u^2} H_{2m}^{(1)}(2zu)u^{2k} du,
\]

where the path of integration runs from \(-\infty\) to \(\infty\) and passes above the singularity at the origin. If we substitute (2.1) for \( H_{2m}^{(1)} \), we obtain

\[
W_{k,m}(z^2) = 2^{1/4-k} \sqrt{z} \left\{ \sum_{r=0}^{m-1} \frac{(2m+r)}{(2z\sqrt{2})} D_{2k-r-1/2}(z\sqrt{2}) + E_k(z) \right\}
\]

where the remainder is given by
\begin{equation}
E_p(z) = \frac{1}{\sqrt{\pi}} 2^{k-1/4} \int_{-\infty}^{\infty} e^{-u^2} e^{u^2 (2k-1/2) R_p(1)} (2zu) \, du.
\end{equation}

This result is well known [4, p. 124]. When \(k\) and \(m\) are fixed, it was shown in [11, (3.1)] that \(E_p(z) = O(e^{-\pi/2 z^{2k-2p-1/2}})\), uniformly in \(|\arg z| \leq \pi/4 - \Delta\). When \(k\) and \(m\) are functions of \(z\), we have the following lemma.

**Lemma 3.** Let \(k\) and \(m\) be real-valued functions of \(z\) for which \(k = o(z)\) and \(m = o(z^{1/2})\) as \(|z| \to \infty\). If \(|m| \geq \delta > 0\) then

\begin{equation}
E_p(z) = O\left(2^{k-1/2} e^{-\pi/2 (m/z)^2}\right).
\end{equation}

If \(|m| \leq \delta\) then

\begin{equation}
E_p(z) = O\left(2^{k-1/2} e^{-\pi/2 (m/z)^2}\right).
\end{equation}

Both results hold uniformly in \(|\arg z| \leq \pi/2 - \Delta\), and the constants implied in \(O\)-symbols are independent of \(k\), \(m\), and \(z\).

**Proof.** Returning to (3.3), we let

\begin{equation}
I = \int_{-\infty}^{\infty} e^{-u^2} e^{u^2 (2k-1/2) R_p(1)} (2zu) \, du.
\end{equation}

In [11], it was shown that by a change of variable \(u = zu'\) followed by a deformation of the contour,

\begin{equation}
I = z^{2k+1/2} \int_{-\infty}^{\infty} e^{-z^2 (x + i)^{2k-1/2} R_p(1)} (2z (x + i)) \, dx,
\end{equation}

the path of integration now being a straight line joining \(-\infty\) to \(\infty\). By Lemma 1,

\begin{equation}
|I| \leq A_p \|(2m, p)| \left| e^{-z^2 z^{2k-2p+1/2}} \right| J,
\end{equation}

where

\begin{equation}
J = \int_{-\infty}^{\infty} \left| e^{-z^2 x^2} (x + i)^{k-p-1/2} \right| dx
\end{equation}

and the constant \(A_p\) depends only on \(p\). Since \(x\) is real, we have \(|x + i| \geq 1\), and so

\begin{equation}
J \leq 2 \int_{0}^{\infty} e^{-\pi x^2} (x^2 + 1)^k \, dx.
\end{equation}

We consider separately the cases \(k \leq 0\) and \(k > 0\).

When \(k \leq 0\),

\begin{equation}
J \leq 2 \int_{0}^{\infty} e^{-(\pi x^2)^{1/2}} \, dx = \left(\frac{\pi}{\text{Re} z^2}\right)^{1/2}.
\end{equation}

Hence, \(J = O(z^{-1})\) for \(z\) restricted to \(|\arg z| \leq \pi/4 - \Delta\).

When \(k > 0\),

\begin{equation}
J \leq 2 \int_{0}^{\infty} e^{-(\pi x^2-k x^2)} \, dx
\end{equation}

provided that the integral exists. Since \(k = o(z)\) as \(|z| \to \infty\),
for sufficiently large $z$ in the sector $|\arg z| \leq \pi/4 - \Delta$, where $\eta_k$ is a positive finite number and independent of $|z|$. Therefore, we again have $J = O(z^{-1})$, as $z \to \infty$ in $|\arg z| \leq \pi/4 - \Delta$.

We have thus proved that a constant $A'_p$ exists such that

$$|I| \leq A'_p |(2m, p)e^{-x^{1/2}z^{2k-2p-1/2}}|,$$

for large values of $z$ in $|\arg z| \leq \pi/4 - \Delta$. The region of validity can be extended to $|\arg z| \leq \pi/2 - \Delta$ by a standard argument. We rotate the path of integration in (3.7) through an arbitrary angle $\gamma$, where $-\pi/4 < \gamma < \pi/4$. When $z$ is positive, use of Cauchy's theorem easily shows that (3.7) is valid if the upper and lower limits are replaced by $\infty e^{i\gamma}$ and $-\infty e^{i\gamma}$ respectively. With this change, (3.7) holds when $|\arg (ze^{i\gamma})| \leq \pi/4 - \Delta$. A repetition of the proof (with some slight modifications) then shows that (3.14) is also valid in this angle. By varying $\gamma$, it follows that (3.14) holds when $|\arg z| \leq \pi/2 - \Delta$.

Since $E_a(z) = (1/\sqrt{\pi})2^{k-1/4}e^{(1/4-k)\gamma z}I$, by (3.14),

$$E_a(z) = O\left(2^{(2m, p)e^{-x^{1/2}z^{2k-2p-1/2}}}\right)$$

for all large values of $z$ restricted to the sector $|\arg z| \leq \pi/2 - \Delta$. When $|m| \leq \delta$, (3.15) is certainly equivalent to (3.5). When $|m| \geq \delta > 0$, (3.4) follows from (3.15) in view of the fact that $(2m, p) \sim (2m^{2/3})$.

**Main Theorem.** Let $k$ and $m$ be real-valued functions of $z$ satisfying conditions (1.1) and (1.2). Then, for any $N \geq 0$,

$$2^{k-1/4}W_{k,m}(z) = \frac{D_{2k-1/2}(2z)^{1/2}}{z^{1/4}} \left[ \sum_{s=0}^{N+1} a_s \frac{z^s}{z^s} + o\left(\left(\frac{m^2}{z}\right)^{2N+2}\right) \right]$$

$$+ \frac{D_{2k-1/2}(2z)^{1/2}}{z^{1/4}} \left[ \sum_{s=0}^{N+1} b_s \frac{z^s}{z^s} + o\left(\left(\frac{m^2}{z}\right)^{2N+2}\right) \right]$$

as $z \to \infty$ in $|\arg z| \leq \pi - \delta$, uniformly with respect to $\arg z$. The coefficients $a_s$ and $b_s$ depend on $k$ and $m$, and are explicitly given in (3.24).

**Proof.** Clearly, $|(m^2/z)^{2s}|$ is an asymptotic sequence under the hypothesis $m = o(z^{1/2})$ as $|z| \to \infty$. Let $N$ be an arbitrary but fixed positive integer, and set

$$S = \sum_{r=0}^{2N+2} \frac{(2m, r)}{(2z)^{1/2}} D_{2k-r-1/2}(2z)^{1/2}.$$ 

The following lemma is given in [10].

**Lemma.** For each $r \geq 0$ we have

$$(-1)^r(-\lambda, D_{-r})(z) = D_{-r}(z)P_r(z) + D_{-r}(z)Q_{-r-1}(z)$$

where $P_r(z)$ and $Q_{-r-1}(z)$ are polynomials of the form

$$P_r(z) = \sum_{s=0}^{(r/2)} p_{r,s}z^{r-2s},$$

$$Q_{-r-1}(z) = \sum_{s=0}^{(r-1)/2} q_{r-1,s}z^{r-(2s+1)}.$$
The coefficients $p_{r,s}$ and $q_{r-1,s}$ can be successively determined from the recurrence relations

\begin{align}
P_{r+1}(z) &= zP_r(z) + (-\lambda + r - 1)P_{r-1}(z), \\
Q_r(z) &= zQ_{r-1}(z) + (-\lambda + r - 1)Q_{r-2}(z),
\end{align}

with $P_0(z) = 1$, $P_1(z) = z/2$, $Q_0(z) = 0$ and $Q_1(z) = 1$.

Now, let $|k| \geq N + 1$ so that $2k - \frac{1}{2} \neq 0, 1, \ldots, 2N + 1$, and hence $(\frac{1}{2} - 2k)$, $\neq 0$ for $r = 0, 1, \ldots, 2N + 2$. It follows from (3.17) that the sum $S$ can be rearranged in the form

\begin{equation}
S = D_{2k-1/2}((2z)^{1/2}) \sum_{s=0}^{N+1} \frac{a_s}{z^s} + D'_{2k-1/2}((2z)^{1/2}) \sum_{s=0}^{N} \frac{b_s}{z^{s+1/2}}
\end{equation}

where

\begin{equation}
a_s = \frac{1}{2^r} \sum_{r \geq 2k} \frac{(-1)^r(2m, r)}{2^r(\frac{1}{2} - 2k)} p_{r,s} \quad \text{and} \quad b_s = \frac{1}{2^{s+1/2}} \sum_{r \geq 2s+1} \frac{(-1)^r(2m, r)}{2^r(\frac{1}{2} - 2k)} q_{r-1,s}.
\end{equation}

Therefore

\begin{equation}
W_{k,m}(z) = 2^{1/4-k} \sqrt{\frac{z}{4}} \left\{ D_{2k-1/2}((2z)^{1/2}) \sum_{s=0}^{N+1} \frac{a_s}{z^s} + D'_{2k-1/2}((2z)^{1/2}) \sum_{s=0}^{N} \frac{b_s}{z^{s+1/2}} + E_{2N+3}(\sqrt{z}) \right\}
\end{equation}

for any fixed integer $N \geq 0$.

Now, it only remains to consider the remainder $E_{2N+3}$. By Lemmas 2 and 3, we have

\begin{equation}
E_{2N+3}(\sqrt{z}) = O\left\{ (m^2/z)^{2N+3} D_{2k-1/2}((2z)^{1/2}) \right\},
\end{equation}

and, similarly,

\begin{equation}
E_{2N+3}(\sqrt{z}) = O\left\{ (m^2/z)^{2N+3} z^{-2N-1/2} D'_{2k-1/2}((2z)^{1/2}) \right\}
\end{equation}

by (3.26). Both results hold uniformly with respect to $\arg z$, as $z \to \infty$ in $|\arg z| \leq \pi - \delta$.

We have thus proved that, for any integer $N \geq 0$,

\begin{equation}
2^{k-1/4} W_{k,m}(z) = \frac{D_{2k-1/2}((2z)^{1/2})}{z^{-1/4}} \left[ \sum_{s=0}^{N+1} \frac{a_s}{z^s} + O\left\{ (m^2/z)^{2N+3} \right\} \right] \\
+ \frac{D'_{2k-1/2}((2z)^{1/2})}{z^{1/4}} \left[ \sum_{s=0}^{N} \frac{b_s}{z^s} + O\left\{ (m^2/z)^{2N+3} \right\} \right],
\end{equation}

as $z \to \infty$ in $|\arg z| \leq \pi - \delta$, uniformly with respect to $\arg z$, which certainly implies the required result.