Multi-Dimensional Extensions of the Chebyshev Polynomials

By Richard O. Hays

Abstract. Two families of polynomials are introduced which satisfy multi-dimensional (or multi-indexed) recursion relationships. These polynomials are developed from the Chebyshev polynomials. Also two additional polynomials are presented which satisfy a special two-dimensional recursion relationship.

I. Introduction. The Chebyshev polynomials belong to the set of ultraspherical or Gegenbauer polynomials and are related to the hypergeometric functions [1]. These polynomials have proven useful in such areas as lattice dynamics [2], numerical analysis [1], and differential equations [1], [3].

The Chebyshev polynomials appear in the literature in various forms, so the following relationships define the forms of the polynomials which will be employed herein [1]:

\[ \frac{1 - x^2}{1 - 2ax + x^2} = T(0; \alpha) + 2 \sum_{n=1}^{\infty} T(n; \alpha)x^n, \]

\[ \frac{1}{1 - 2ax + x^2} = \sum_{n=0}^{\infty} U(n; \alpha)x^n, \]

where \( T(n; \alpha) \) and \( U(n; \alpha) \) are the Chebyshev polynomials of the first and second kind, respectively, and \( T(0; \alpha) = 1 \).

The terms \((1 - x^2)/(1 - 2ax + x^2)\) and \(1/(1 - 2ax + x^2)\) are the generating functions for the Chebyshev polynomials of the first and second kind, respectively, where the expressions (1.1) and (1.2) are valid, provided \(|x| < \min |\alpha \pm (\alpha^2 - 1)^{1/2}|\).

The expressions for \( T(n; \alpha) \) and \( U(n; \alpha) \) are

\[ T(n; \alpha) = \frac{\eta}{2} \sum_{m=0}^{\lfloor n/2 \rfloor} (-1)^m \frac{\eta^{n-m-1}!}{m! (n - 2m)!} (2\alpha)^{n-2m}, \]

\[ U(n; \alpha) = \sum_{m=0}^{\lfloor n/2 \rfloor} (-1)^m \frac{n-m-1}! \frac{n-m!}{m! (n - 2m)!} (2\alpha)^{n-2m}. \]

Let \( T(n; \alpha) \) represent either \( T(n; \alpha) \) or \( U(n; \alpha) \); then \( I(n; \alpha) \) satisfies the recursion relationship

\[ 2\alpha I(n + 1; \alpha) - I(n + 2; \alpha) - I(n; \alpha) = 0. \]

II. Extensions to Two Dimensions. The Chebyshev polynomials can be extended to two dimensions by forming multivariate generating functions produced...
by replacing $\alpha$ by $(\alpha - (y + y^{-1})/2)$ in the original generating functions. Employing
the multinomial theorem, we find that

\begin{equation}
T(n; \alpha - (y + y^{-1})/2) = \sum_{r=0}^{n} T(n; r; \alpha)y^r,
\end{equation}

\begin{equation}
U(n; \alpha - (y + y^{-1})/2) = \sum_{r=0}^{n} U(n; r; \alpha)y^r,
\end{equation}

where

\begin{equation}
T(n; r; \alpha) = \frac{n}{2} \sum_{m=0}^{\lfloor n/2 \rfloor} \frac{(-1)^{m+r}(n - m - 1)!}{m!} \sum_{k=0}^{\lfloor r/2 \rfloor} \frac{K(2\alpha)^k H(q)}{q!},
\end{equation}

\begin{equation}
U(n; r; \alpha) = \sum_{m=0}^{\lfloor n/2 \rfloor} \frac{(-1)^{m+r}(n - m)!}{m!} \sum_{k=0}^{\lfloor r/2 \rfloor} \frac{K(2\alpha)^k H(q)}{q!},
\end{equation}

subject to the relations

$$
\beta = (n - |r| - 2m)/2,
$$

$$
K = 1/k! (k + |r|)!,
$$

$$
q = n - |r| - 2m - 2k,
$$

and where $H(q)$ is the Heaviside step function,

$$
H(q) = \begin{cases} 
0 & \text{if } q < 0 \\
1 & \text{if } q \geq 0 
\end{cases}
$$

$I(n; r; \alpha)$ satisfies the recursion relationship

\begin{equation}
2\alpha I(n + 1; r + 1; \alpha) - I(n + 2; r + 1; \alpha) - I(n; r + 1; \alpha) - I(n + 1; r + 2; \alpha) - I(n + 1; r; \alpha) = 0,
\end{equation}

where $I(n; r; \alpha)$ represents either $T(n; r; \alpha)$ or $U(n; r; \alpha)$.

Several of the $U(n; r; \alpha)$ polynomials are displayed in Table I.

<table>
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<tr>
<th>$U(n; r; \alpha)$ Polynomials</th>
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III. Extensions to $N + 1$ Dimensions. The generalization to $N + 1$ dimensions is straightforward with the replacement of $\alpha$ by

$$\left( \alpha - \frac{y_1 + y_1^{-1} + y_2 + y_2^{-1} + \cdots + y_N + y_N^{-1}}{2} \right)$$

in the generating functions for the original Chebyshev polynomials. $T$ and $U$ are given by

$$T(n; r_1, r_2, \ldots, r_N; \alpha)$$

$$U(n; r_1, r_2, \ldots, r_N; \alpha)$$

with

$$\gamma = r_1 + r_2 + \cdots + r_N,$$

$$\beta_p = (n - |r_1| - |r_2| - \cdots - |r_p|) - 2m - 2k_1 - 2k_2 - \cdots - 2k_{p-1})/2$$

for $p = 1, 2, 3, \ldots, N$, if we define $k_0 = 0$,

$$K_p = 1/k_p! \ (k_p + |r_p|)! \quad p = 1, 2, 3, \ldots, N,$$

$$q = 2(\beta_N - k_N).$$

With mathematical induction, we find that $I(n; r_1, r_2, \ldots, r_N; \alpha)$ satisfies the recursion relationship

$$\sum_{k=0}^{N} \sum_{k=0}^{m} C(M_k, N)I(t_k; S_{1,k}, S_{2,k}, \ldots, S_{N,k}; \alpha) = 0,$$

where

$$C(M_k, N) = \begin{cases} 
\frac{2\alpha}{N + 1} & \text{if } M_k = 0 \\
-1 & \text{if } M_k = -1, 1
\end{cases}$$

$$t_k = n + 1 + M_0 \delta_{k,0},$$

$$S_{a,k} = r_a + 1 + M_a \delta_{a,k},$$

$$\delta_{k,a} = \begin{cases} 
1 & \text{if } k = a \\
0 & \text{if } k \neq a
\end{cases}$$

and $I(t_k; S_{1,k}, S_{2,k}, \ldots, S_{N,k}; \alpha)$ represents either $T(t_k; S_{1,k}, S_{2,k}, \ldots, S_{N,k}; \alpha)$ or $U(t_k; S_{1,k}, S_{2,k}, \ldots, S_{N,k}; \alpha)$. 

IV. Special Two-Dimensional Polynomials. Sometimes, recursion relationships arise which are similar to Eq. (2.5) but differing in the coefficients of the $I'$s.
Consider the recursion relationship

\[ 2αI(n + 1; r + 1; β, γ; α) - βI(n + 2; r + 1; β, γ; α) - \gamma I(n + 1; r + 1; β, γ; α) = 0. \]

(4.1)

An extension of the Chebyshev polynomials allows for the determination of the polynomials which satisfy Eq. (4.1).

Replacing \( α \) by

\[ \left[ \frac{α}{β} - \frac{γ}{2β} (y + y^{-1}) \right] \]

in the generating functions produces the polynomials

\[ T(n; r; β, γ; α) = \sum_{m=0}^{[n/2]} \frac{(-1)^{m+r}(n - m - 1)!}{m!} \left( \frac{γ}{β} \right)^{n-2m} \sum_{k=0}^{[β/2]} \frac{K(2α/γ)^{γ}H(q)}{q!}, \]

(4.2)

\[ U(n; r; β, γ; α) = \sum_{m=0}^{[n/2]} \frac{(-1)^{m+r}(n - m)!}{m!} \left( \frac{γ}{β} \right)^{n-2m} \sum_{k=0}^{[β/2]} \frac{K(2α/γ)^{γ}H(q)}{q!}, \]

(4.3)

where \( β, K, q, \) and \( H \) are the same as for Eq. (2.4).

The \( T \) and \( U \) polynomials of Eqs. (4.2) and (4.3) satisfy Eq. (4.1).

V. Comments. A solution to Eq. (1.5), where \( I \) does not necessarily represent \( T \) or \( U \), can be written in terms of the Chebyshev polynomials. It appears that solutions to the higher-order recursion relationships should consist of combinations of the extended Chebyshev polynomials.

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