A First Order Method for Differential Equations of Neutral Type

By R. N. Castleton and L. J. Grimm*

Abstract. A first order method is presented for solution of the initial-value problem for a differential equation of neutral type with implicit delay in the critical case where the time-lag is zero and the method of stepwise integration does not apply. A convergence theorem is proved, and numerical examples are given.

1. Introduction. In this note, we present a first order method for the numerical solution of the initial-value problem (IVP) for a neutral-type functional-differential equation without previous history:

(1) \[ x'(t) = f(t, x(t), x(g(t, x(t))), x'(g(t, x(t)))) \]
(2) \[ x(a) = x_0, \quad x'(a) = z_0, \]

where \( z_0 \) is a real root of the algebraic equation

(3) \[ z = f(a, x_0, x_0, z). \]

Here, \( x(t) \) is a scalar function to be determined on some finite interval \([a, b]\). We shall make the following assumptions regarding \( f \) and \( g \):

(H1) \( f \) and \( g \) are continuous and satisfy uniform Lipschitz conditions of the form

\[ |f(t, x_1, y_1, z_1) - f(t, x_2, y_2, z_2)| \leq L_1 |x_1 - x_2| + |y_1 - y_2| + L_2 |z_1 - z_2|, \]
\[ |g(t, x_1) - g(t, x_2)| \leq L_3 |x_1 - x_2| \]

in their respective domains \( E \) and \( E' \), where

\[ E = \{(t, x, y, z): a \leq t \leq b, |x - x_0| \leq c, |y - x_0| \leq c, |z| \leq M\} \]

and \( E' \) is the projection of \( E \) in the \((t, x)\) space; \( c, M, L, L_1, L_2 \) are constants, with \( L_2 < 1, M \) is such that \( \sup_{(t, x, y, z) \in E} |f(t, x, y, z)| < M \), and \( M(b - a) < c \).

(H2) \( a \leq g(t, x) \leq t \) for \((t, x) \in E'\).

Our hypotheses, together with additional smoothness and growth conditions on \( f \) and \( g \), ensure the local existence of a solution of the IVP (1)–(2). Furthermore, \( x(t) \) is the only solution having a bounded derivative on \([a, b]\); see [2], [4]. Our result extends a method developed by Feldstein [3] for the equation of retarded type

Received August 7, 1970.

AMS (MOS) subject classifications (1970). Primary 34K99; Secondary 65L05.

Key words and phrases. Equations of neutral type, functional-differential equations, implicit-delay equations, numerical methods.

* Research of second author supported by National Science Foundation.

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to the neutral-type equation with implicit delay (1). Other methods for implicit-delay equations are given in [1].

2. The Algorithm \( \mathfrak{A} \). Let \( y(t) = x(g(t, x(t))) \); \( z(t) = x'(g(t, x(t))) \). Let \( N \) be a positive integer, and let \( h = (b - a)/N \). For each nonnegative integer \( n \leq N \), let \( t_n = a + nh \). Let \([s]\) denote the integer part of \( s \). Define the algorithm \( \mathfrak{A} \) as follows:

\[
\begin{align*}
&f_n = f(t_n, x_n, y_n, z_n), &g_n = g(t_n, x_n), \\
&q(n) = \lfloor(g_n - a)/h\rfloor, &r(n) = (g_n - a)/h - q(n), \\
&y_0 = x_0, &y_n = x_{q(n)} + hr(n)f_{q(n)}, \\
&z_n = f_{q(n)}, \\
&x_{n+1} = x_n + hf_n.
\end{align*}
\]

Note that condition (H2) implies \( q(n) \leq n \), thus, the algorithm is well defined. For \( n = 0 \), \( g_0 = a \), \( q(0) = 0 \), and \( r(0) = 0 \). Thus, \( y_0 = x_0 \) and \( z_0 = f(a, x_0, x_0, z_0) \). Let \( u_0 \), an approximation of the root \( z_0 \), be chosen independently of \( h \). It is of interest to note that such an approximation does not destroy the order \( h \) convergence of the algorithm. It is of further interest that (6) may be simplified to \( y_n = x_{q(n)} \). The error bound established in the convergence theorem for this "simplified" algorithm is larger but still of order \( h \), as noted following the proof of convergence of the algorithm \( \mathfrak{A} \). The second numerical example of Section 4 demonstrates both the algorithm \( \mathfrak{A} \) and the simplified algorithm.

If \( g_n = t_n \) for any \( n \), \( 1 \leq n \leq N \), then \( q(n) = n \), \( r(n) = 0 \), and (7) becomes \( z_n = f(t_n, x_n, y_n, z_n) \) which has exactly one root \( z \) in the interval \([-M, M]\) under the conditions (H1)-(H2) together with the smoothness and growth conditions mentioned in Section 1. We must in general include a procedure for finding this root, and this in turn will affect the error estimate. As before, such an estimate does not destroy the order \( h \) convergence of the algorithm. For simplicity, we do not take this into account, since our aim is to show the convergence of the algorithm \( \mathfrak{A} \).

Thus, we shall assume in the convergence proof that (7) will not reduce to \( z_n = f(t_n, x_n, y_n, z_n) \), \( n \geq 1 \).

3. Convergence.

Theorem. Let \( f \) and \( g \) satisfy (H1)-(H2) and suppose, in addition, that there exists a unique solution \( x(t) \) of (1)-(2) with \( \sup_{[a, b]} |x''(t)| \leq B \). Then, for each \( t_n \in [a, b] \), \( 0 < n \leq N \),

\[
|x_n - x(t_n)| \leq h\left(L_s |z_0 - u_0| e^{s(h-a)} + \frac{B}{2s} \frac{1}{1 - L_s} \left(e^{s(h-a)} - 1 \right) \right) + O(h^2)
\]

where

\[
s = L(1 + c_o) + L_c e_1,
\]
\[ c_0 = 1 + M L_0, \]
\[ c_1 = (L(2 + M L_0) + B L_0)/(1 - L), \]

\( u_0 \) is the approximation to \( z_0 \) mentioned above, and \( x_n \) is given by algorithm A.

**Proof.** Let \( e_n = |x_n - x(t_n)| \); \( e_n^* = |y_n - y(t_n)| \); \( e_n^{**} = |z_n - z(t_n)| \). From (8) and Taylor's formula, we obtain

\[ e_{n+1} \leq e_n + h(L(e_n + e_n^*) + L e_n^{**}) + h^2 B/2. \]

Equation (5) implies that \( g_n = t_{q(n)} + h r(n) \), and hence, in a similar manner, we have (after replacing \( n \) by \( n + 1 \))

\[ e_{n+1}^* \leq M L e_{n+1} + e_{q(n+1)} \]
\[ + h(n + 1) [L e_{q(n+1)} + e_{q(n+1)}^* + L e_{q(n+1)}^{**}] + h^2 r(n + 1)/2, \]
\[ e_{n+1}^{**} \leq B L e_{n+1} + L e_{q(n+1)} + e_{q(n+1)}^* + L e_{q(n+1)}^{**} + h r(n + 1)/2. \]

We then have two cases to consider:

**Case 1.** \( q(n + 1) = n + 1 \) and \( r(n + 1) = 0 \). Under these conditions, (9) is unchanged:

\[ e_{n+1} \leq e_n(1 + hL) + e_n^* hL + e_n^{**} hL + h^2 B/2. \]

(10) becomes

\[ e_{n+1}^* \leq e_{n+1}(1 + ML) = e_{n+1} c_0. \]

And (11) becomes

\[ e_{n+1}^{**} \leq (L + BL) e_{n+1} + Le_{n+1} + L e_{**} \]

or

\[ e_{n+1}^{**} \leq \left( L + BL + L(1 + ML) \right) e_{n+1} = e_{n+1} c_0. \]

Define the partial ordering for vectors: \( v_1 = (v_1^1, \ldots, v_1^k) \) \( \leq v_2 = (v_2^1, \ldots, v_2^k) \) if \( v_1^i \leq v_2^i \), \( i = 1, \ldots, k \). Then, in vector form, (9a), (10a), and (11a) become

\[
\begin{bmatrix}
  e_{n+1}^* \\
  e_{n+1}^* \\
  e_{n+1}^{**}
\end{bmatrix}
\leq
\begin{bmatrix}
  1 + hL & hL & hL \\
  (1 + hL)c_0 & hLc_0 & hLc_0 \\
  (1 + hL)c_1 & hLc_1 & hLc_1
\end{bmatrix}
\begin{bmatrix}
  e_n \\
  e_n^* \\
  e_n^{**}
\end{bmatrix}
+ \begin{bmatrix}
  h/2 \\
  hB \\
  hc_0/2
\end{bmatrix}
\]

which is of the form \( d_{n+1} \leq A d_n + b_1 \).

**Case 2.** \( q(n + 1) \leq n \) and \( 0 \leq r(n + 1) < 1 \).

Let

\[ \delta_n = \max_{1 \leq i \leq n} e_i, \quad \delta_n^* = \max_{1 \leq i \leq n} e_i^*, \quad \delta_n^{**} = \max_{1 \leq i \leq n} e_i^{**}. \]

Then, (9) becomes

\[ \delta_{n+1} \leq \delta_n(1 + hL) + \delta_n^* hL + \delta_n^{**} hL + h^2 B/2. \]
And (10) becomes
\[ \delta_{n+1}^* \leq ML_n \delta_{n+1} + \delta_n(1 + hL) + hL \delta_n^* + hL \delta_{n+1}^* + h^2 B/2. \]

Using (9b), we have
\[ \delta_{n+1}^* \leq (\delta_n(1 + hL) + \delta_n^* hL + \delta_n^{**} hL^2 + h^2 B/2)(1 + ML_n) \]
or
\[ (10b) \quad \delta_{n+1}^* \leq \delta_n(1 + hL) + \delta_n^* hL + \delta_n^{**} hL_0 + \delta_n^{*} hL_0 + h^2 c_0 B/2. \]

Finally, (11) becomes
\[ \delta_{n+1}^{**} \leq \delta_{n+1} B L_n + \delta_n L + \delta_n^{*} L + \delta_n^{**} L + hB. \]

Further, enlarging \( \delta_n \) to \( \delta_{n+1} \) and \( \delta_n^* \) to \( \delta_{n+1}^* \) on the right, and using \( 1 - L > 0 \), we find
\[ \delta_{n+1}^{**} \leq \delta_{n+1}(L + B L_n) + \delta_{n+1} \frac{L}{1 - L} + \frac{hB}{1 - L}. \]

Using (9b) and (10b), we have
\[ \delta_{n+1}^{**} \leq \left( \frac{L + B L_n}{1 - L} \right) \left( \delta_n(1 + hL) + \delta_n^* hL + \delta_n^{**} hL + \frac{h^2 B}{2} \right) + \frac{hB}{1 - L}, \]
or
\[ (11b) \quad \delta_{n+1}^{**} \leq \delta_n(1 + hL) + \delta_n^* hL + \delta_n^{**} hL_{c_0} + \delta_n^{*} hL_{c_0} + \frac{hB}{1 - L} + \frac{h^2 c_0 B}{2}. \]

Then, as a vector system, (9b), (10b), and (11b) become
\[
\begin{bmatrix}
\delta_{n+1} \\
\delta_{n+1}^* \\
\delta_{n+1}^{**}
\end{bmatrix}
\leq
\begin{bmatrix}
1 + hL & hL & hL^2 \\
(1 + hL)c_0 & hLc_0 & hLc_0 \\
(1 + hL)c_0 & hLc_0 & hLc_0
\end{bmatrix}
\begin{bmatrix}
\delta_n \\
\delta_n^* \\
\delta_n^{**}
\end{bmatrix}
+ \begin{bmatrix}
h/2 \\
hBc_0/2 \\
hBc_0/2 + 1/(1 - L)
\end{bmatrix}
\]
which is of the form \( d_{n+1} \leq A d_n + b \). Comparing this with the result obtained in Case 1, we find that \( A_1 \) and \( A_2 \) are identical and that \( b_1 \leq b_2 \). Thus, any bound obtained here in Case 2 for \( d_{n+1} \) will also bound \( d_{n+1} \) in Case 1.

To complete the proof, we shall use the following lemmas [3] which may be verified by induction:

**Lemma 1.** Suppose \( A \) is a \( k \times k \) real matrix and \( b \) is a real \( k \)-vector. Let \( \{d_n\} \) \( (n = 0, 1, \ldots) \) satisfy \( d_{n+1} \leq A d_n + b \). Then
\[
d_{n+1} \leq A^{n+1} d_0 + \left( \sum_{i=0}^{n} A^i \right) b.
\]

**Lemma 2.** Let \( p = (p_1, \ldots, p_k) \), \( q = (q_1, \ldots, q_k) \). Suppose the \( k \times k \) matrix A has the form \( A = p^T q \). Then
\[
A^n = \left( \sum_{i=1}^{k} p_i q_i \right)^{n-1} A.
\]
By Lemma 1,
\[ d_{n+1} \leq A_2^{n+1} d_0 + \left( \sum_{i=0}^{n} A_i \right) b_2, \]
where
\[ d_0 = \begin{bmatrix} c_0 \\ e_0^n \\ e_0^{n*} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \]

Then, because
\[ A_2 = \begin{bmatrix} 1 \\ c_0 \end{bmatrix} (1 + hL, hL, hL), \]
we can make use of Lemma 2 to obtain
\[ A_2' = (1 + hL + hLc_0 + hLc_1)^{-1} A_2 = (1 + hs)^{-1} A_2. \]

Two results follow from this: \[ A_2^{n+1} = (1 + hs)^n A_2 \leq e^{e(x(b-a))} A_2, \]
and
\[ \sum_{i=1}^{n} A_i = A_2 \sum_{i=1}^{n} (1 + hs)^{i-1} = \frac{(1 + hs)^n - 1}{hs} A_2 \leq \frac{1}{hs} (\exp(s(b-a)) - 1) A_2. \]

Finally,
\[ d_{n+1} \leq A_2^{n+1} d_0 + \left( \sum_{i=0}^{n} A_i \right) b_2 \]
\[ \leq h \left| z_0 - u_0 \right| L_i e^{e(x(b-a))} \begin{bmatrix} 1 \\ c_0 \\ c_1 \end{bmatrix} \]
\[ + \frac{B}{2s} \left( hs + \frac{1 + L_i}{1 - L_i} \right) (e^{e(x(b-a))} - 1) \begin{bmatrix} 1 \\ c_0 \\ c_1 \end{bmatrix} + B \begin{bmatrix} \frac{h}{2} \\ \frac{hc_0}{2} \\ \frac{hc_1}{2} + \frac{1}{1 - L_i} \end{bmatrix} \]

which gives
\[ c_{n+1} \leq \delta_{n+1} \leq h \left( \left| z_0 - u_0 \right| L_i e^{e(x(b-a))} + \frac{B}{2s} \left( hs + \frac{1 + L_i}{1 - L_i} (e^{e(x(b-a))} - 1) + \frac{hB}{2} \right) \right) \]
and the theorem follows.
For the simplified algorithm, where \( y_n = x_{q(n)} \) the following bound is possible:

\[
d_{n+1} \leq h \begin{bmatrix}
|z_0 - u_0| L_i e^{s(b-a)} \\
|c_0| \\
|c_1|
\end{bmatrix} + \left( B + \left( \frac{h}{2s} \left( h_i + \frac{1}{1 - L_i} \right) \frac{ML}{1 - L_i} \right) e^{s(b-a)} - 1 \right) \begin{bmatrix}
1 \\
|c_0| \\
|c_1|
\end{bmatrix} + B \begin{bmatrix}
\frac{h}{2} \\
\\frac{hc_0}{2} \\
\frac{hc_1}{2} + \frac{1}{1 - L_i}
\end{bmatrix} + \begin{bmatrix}
0 \\
\frac{M}{1 - L_i}
\end{bmatrix},
\]

and hence

\[
e_{n+1} \leq h \begin{bmatrix}
|z_0 - u_0| L_i e^{s(b-a)} \\
|c_0| \\
|c_1|
\end{bmatrix} + \left( B + \left( \frac{h}{2s} \left( h_i + \frac{1}{1 - L_i} \right) \frac{ML}{1 - L_i} \right) e^{s(b-a)} - 1 \right) \begin{bmatrix}
1 \\
|c_0| \\
|c_1|
\end{bmatrix} + hB
\]

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Table II. \( x_n^{(1)}(h) \) denotes the value of \( x_n \) for step size \( h \) by algorithm \#. \( x_n^{(2)}(h) \) denotes the value of \( x_n \) for step size \( h \) by the simplified algorithm.

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4. Examples. (a) We solve the IVP

\[
x'(t) = \frac{-4tx^2(t)}{4 + \log^2 \cos t} + \tan 2t + \frac{1}{2} \tan^{-1} z
\]

\((z_0 = 0, x_0 = 0, z = x'(g(t, x(t))) = x'(tx^2(t))/(1 + x^2(t)))\) on the interval \([0, .75]\). The existence and uniqueness of the solution is guaranteed by the results of [2] mentioned earlier. The only solution is \( x(t) = -\frac{1}{2} \log \cos 2t \).

The results of the computation by algorithm \( \# \) are given in Table I.

(b) Consider the IVP

\[
x''(t) = \cos t(1 + y) + xz - \sin(t(1 + \sin^2 t)),
\]

with \( y = x(t^2(t)), z = x'(tx^2(t)) \), \( z_0 = 1, x_0 = 0 \), on the interval \([0, 1]\). As in example (a), existence and uniqueness of the solution are guaranteed by the results of [2]. Here, the solution is \( x(t) = \sin t \).

The results of the computation by the algorithm \( \# \) and by the simplified algorithm are given in Table II.