An Elliptic Integral Identity

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Abstract. The identity

\[ K(\tau) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp \left[ -\frac{1}{2}(x^4 - 2(2\tau^2 - 1)x^2y^2 + y^4) \right] dx \, dy, \]

where \( K(\tau) \) is the complete elliptic integral of the first kind, is used to prove that

\[ K(\sqrt{2} - 1) = \pi^{1/4}(2 + \sqrt{2})^{1/4}/4\Gamma(\frac{1}{4})\Gamma(\frac{1}{4}). \]

The identity

\[ (1) \quad K(\tau) = -\frac{s^2}{12\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp \left[ -\frac{1}{2}(x^4 - 2(2\tau^2 - 1)x^2y^2 + y^4) \right] dx \, dy, \quad |\tau| < 1, \]

is easily proved by using the standard transformation \( x = r \cos v, \; y = r \sin v \). Before proceeding, we will list a few identities for the benefit of the reader.

\[ (2) \quad \int_0^\infty e^{-pt} t^{-1/2} e^{-t^{1/4}a} \, dt = \alpha^{-1/2} p^{-1/2} e^{\alpha^{1/2}} K_{1/4}(\alpha p), \quad \text{Re} \, \alpha > 0 \quad [1, \text{p. 146}], \]

\[ (3) \quad \int_0^\infty e^{-pt} t^{-1/2} K_{\nu + 1/2}(\alpha t) \, dt = 2^{-1/2} \alpha^{-1/2} \pi^{-1/2} \Gamma(\mu + \nu) \Gamma(\mu + \nu + 1) e^{-p/\alpha} P_{\nu}(p/\alpha), \]

\[ s = (p^2 - \alpha^2)^{1/2}, \quad \text{Re} \, (\mu + \nu) > -1, \quad \text{Re} \, (\mu - \nu) > 0, \]

\[ \text{Re} \, p > -\text{Re} \, \alpha, \quad [1, \text{p. 198}], \]

\[ (4) \quad P_p^s(0) = \frac{2^{s-1/2}}{\Gamma(1/2(p - q) + 1)\Gamma(1/2(-p - q + 1))} \quad [2, \text{p. 354}]. \]

We will now calculate the integral (1) in a different way.

Integration with respect to \( x \) yields

\[ \int_{-\infty}^{\infty} \exp \left[ -\frac{1}{2}(x^4 - 2(2\tau^2 - 1)x^2y^2) \right] dx = \int_0^{\infty} t^{-1/2} e^{-t^{1/4}e^{2(\tau^2 - 1)t^{1/2}}} dt \]

\[ = \frac{(1 - 2\tau^2)^{1/2}}{2^{1/2}} |y| \exp \left[ \frac{(2\tau^2 - 1)^2 y^4}{4} \right] K_{1/4} \left( \frac{(2\tau^2 - 1)^2 y^4}{4} \right), \quad 1 - 2\tau^2 > 0. \]

Integration with respect to \( y \) now gives

\[ K(\tau) = \frac{(1 - 2\tau^2)^{1/2}}{2\pi^{1/2}} \int_{-\infty}^{\infty} |y| e^{-y^{1/2}} \exp \left[ \frac{(2\tau^2 - 1)^2 y^4}{4} \right] K_{1/4} \left( \frac{(2\tau^2 - 1)^2 y^4}{4} \right) dy. \]

A few manipulations finally give
Using Eq. (3) we can write

$$K(\tau) = \frac{\pi}{2(1 - 2\tau)^{1/2}} P_{-1/4}\left(\frac{1 + 4\tau^2 - 4\tau^4}{(1 - 2\tau^2)^2}\right).$$

To get (6) in a more convenient form, we use the well-known formula

$$K(i\tau) = \frac{1}{(1 + \tau^2)^{1/2}} K\left(\frac{\tau}{(1 + \tau^2)^{1/2}}\right).$$

Then

$$K\left(\frac{\tau}{(1 + \tau^2)^{1/2}}\right) = \frac{\pi(1 + \tau^2)^{1/2}}{2(1 + 2\tau^2)^{1/2}} P_{-1/4}\left(\frac{1 - 4\tau^2 - 4\tau^4}{(1 + 2\tau^2)^2}\right).$$

Now put $1 - 4\tau^2 - 4\tau^4 = 0$ which yields $\tau^2 = -\frac{1}{2} \pm \frac{1}{2}\sqrt{2}$. A little calculation and Eq. (4) give

$$K(\sqrt{2} - 1) = \frac{\pi^{3/2}(2 + \sqrt{2})^{1/2}}{4\Gamma(\frac{3}{8})\Gamma(\frac{5}{8})}. \quad \text{Q.E.D.}$$

The identity (8) has been proved previously; see for instance [3, pp. 535–536]. The present derivation seems, however, to give a better insight into the problem.

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