Reduction Formulas for Multiple Series

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Abstract. A simple procedure is given for reducing broad classes of multiple series to single series. Examples are given for double series.

Suppose that \( A_1, A_2 = A_3 \), where \( A_i \) is a function of \( u \) and possesses a series expansion \( A_i = \sum_n \phi_i(n, u) \). Then we have

\[
\sum_{m,n} \phi_1(n, u) \phi_2(m, u) = \sum_n \phi_3(n, u).
\]

If both sides of (1) are multiplied by some function \( f(u) \) and integrated over \( u \), we shall have formally

\[
\sum_{m,n} F_1(m, n) = \sum_n F_2(n).
\]

This rather trivial procedure can lead to some remarkable and useful results, as we shall illustrate by some examples.

If \( f \) and \( g \) are two analytic functions, then, upon multiplication of their Taylor series, we obtain

\[
\sum_{m,n=0}^\infty \frac{f^{(m)}(0) g^{(n)}(0)}{m! n!} F(m + n + 1) = \sum_{n=0}^\infty \frac{(fg)^{(n)}(0)}{n!} F(n + 1),
\]

where \( F \) is any Mellin transform.

From the theory of elliptic functions [1], we have the Fourier series

\[
\begin{align*}
(a) \quad & \text{cn}(2Kx/\pi) = (2\pi/kK) \sum_{n=0}^\infty q^{n+1/2}(1 + q^{2n+1})^{-1} \cos(2n + 1)x, \\
(b) \quad & (2K/\pi) \text{dn}(2Kx/\pi) = 1 + 4 \sum_{n=1}^\infty q^n(1 + q^{2n})^{-1} \cos 2nx, \\
(c) \quad & (2K/\pi) \text{cn}(2Kx/\pi) \text{dn}(2Kx/\pi) \\
& = (2\pi/kK) \sum_{n=0}^\infty (2n + 1)q^{n+1/2}(1 - q^{2n+1})^{-1} \cos(2n + 1)x,
\end{align*}
\]

where \( q = e^{-K'/K} \). If we now let \( K'/K = (2u/\pi) \), we find on multiplication of (4(a)) by (4(b)) that

\[
\sum_{m,n=0}^\infty \frac{\cos(2m + 2n + 1)x}{\cosh(2m + 1)u \cosh 2nu} = 2 \sum_{n=0}^\infty \frac{(2n + 1) \cos(2n + 1)x}{\sinh(2n + 1)u},
\]

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where the addition theorem and evenness for the cosine have been used to simplify the left-hand side.

Next, we multiply both sides of (5) by some summable function \( f(x) \), with cosine transform \( F(y) \), and integrate over \( x \) between the limits 0 and \( \infty \). Thus, we have

\[
\sum_{m, n=-\infty}^{\infty} \frac{F(2m + 2n + 1)}{\cosh(2m + 1)u \cosh(2nu)} = 2 \sum_{n=0}^{\infty} \frac{(2n + 1)F(2n + 1)}{\sinh(2n + 1)u}.
\]

This remarkable result is valid for any summable function \( F(x) \).

For example, consider \( F_k(x) = 1 \) for \( (2k + \frac{3}{2}) > |x|, 0 \) otherwise. Denoting the sum on the left-hand side of (6) by \( S_k \), we find that it can be written

\[
S_k = s_0 + s_1 + \cdots + s_k,
\]

where

\[
s_k = 4 \sum_{n=-\infty}^{\infty} [\cosh(4n + 2k + 1)u + \cosh(2k + 1)u]^{-1}.
\]

On the other hand, the sum on the right-hand side of (6) is finite and we have

\[
S_k = 2[\text{csch } u + 3 \text{ csch } 3u + \cdots + (2k + 1) \text{ csch}(2k + 1)u].
\]

Therefore, \( s_k = S_k - S_{k-1} = 2(2k + 1) \text{ csch}(2k + 1)u \) and, hence,

\[
\sum_{n=1}^{\infty} [\cosh(2nu) + \cosh(2k + 1)u]^{-1}
\]

\[
= \frac{1}{2}[(2k + 1) \text{ csch}(2k + 1)u - \frac{1}{2} \text{ sech}^2(k + \frac{1}{2})u], \quad k = 0, 1, 2, \ldots.
\]

In a similar way, we can derive

\[
\sum_{m, n=-\infty}^{\infty} \frac{F(m + n + 1) + F(m - n)}{\sinh(2m + 1)u \cosh(2n + 1)u} = 8 \sum_{n=1}^{\infty} \frac{nF(n)}{\cosh(2nu)},
\]

\[
\sum_{k, m, n=-\infty}^{\infty} \frac{F(k + m + n + 1) + F(k - m + n)}{\cosh(2ku) \cosh(2m + 1)u \sinh(2n + 1)u} = 8 \sum_{n=1}^{\infty} \frac{n^2F(n)}{\sinh(2nu)},
\]

where \( F \) is any sine transform (and hence odd).

Thus, taking \( F(1) = -F(-1) = 1, F(n) = 0, n \neq 1, \) in (12), we obtain the interesting double series

\[
\sum_{k, n=-\infty}^{\infty} \frac{\sinh(2k + n + 1)u}{\cosh(2ku) \sinh(2n + 1)u[\cosh 2(2k + 2n + 1)u + \cosh 4u]} = \text{csch}^2 2u.
\]

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