Four Large Amicable Pairs

By H. J. J. te Riele

Abstract. This note gives a report of systematic computer tests of Euler's rule and several Thabit-ibn-Kurrah-rules, in search of large amicable pairs. The tests have yielded four amicable pairs, which are much larger than the largest amicable pair thus far known.

1. The pair of 25-digit numbers

\[(45222 \ 6553454520 \ 8537974785, \ 45398 \ 0132623392 \ 8286140415)\]

has been the largest known amicable pair since 1946 ([8], [10]). This note gives four new amicable pairs with 32-, 40-, 81-, and 152-digit numbers, as a result of systematic computer tests by Euler's rule (Section 2) and several Thabit-ibn-Kurrah-rules (Sections 3 and 4).

In this research, primality of very large numbers \(N\) had to be established, where \(N + 1\) can be easily factorized; this was done by use of the following:

**Theorem (Lucas-Lehmer [11, p. 442]).** Let \(P\) and \(Q\) be relatively prime integers and let \(U_0 = 0, U_1 = 1, U_{i+1} = PU_i - QU_{i-1}\) for \(i \geq 1\). If \(N\) is a natural number, relatively prime to \(2P - 8Q\), and if \(U_{N+1} \mod N = 0\), while \(U_{(N+1)/p} \mod N \neq 0\) for each prime \(p\) dividing \(N + 1\), then \(N\) is prime.

It is convenient to choose \(P = 1\), while \(Q\) has to be chosen such that \(D^\left(\frac{N-1}{2}\right) \mod N = -1\), where \(D = P - 4Q\).

In the sequel, the indication "\((Q = A)\)" after a number means that primality of that number was established by use of this Lucas-Lehmer theorem, with \(Q = A\). The computations were carried out on the Electrologica-X8 computer of the Mathematical Centre; the value of \(U_i \mod N\) was computed in \(O(\log i)\) steps by use of the binary method (see [6, p. 360 (Exercise 15) and p. 421 (Exercise 26)].

2. Euler's rule [4] for amicable numbers is given by: \(2^npq\) and \(2^nr\) are amicable numbers, if the three integers \(p = 2^{n-m} + 1, q = 2^n + 1\) and \(r = 2^{2n-f} - 1\) are primes, with \(f = 2^n + 1\) and \(n > m \geq 1\). For \(m = 1\), this rule is due to Thabit ibn Kurrah and yields amicable numbers for \(n = 2, 4, 7\), but for no other value \(n \leq 1000\) (see [12, p. 874]). Only one more solution of Euler's rule was known thus far, viz., \(m = 7, n = 8\) (Legendre, Chebyshev).

A systematic computer search for triples \((p, q, r)\) such that both \(p, q\) and \(r\) are primes was carried out for all values of \(n, m\) with \(n > m > 1\) and \(r < 10^{137}\); this search yielded just one new solution, viz., \(m = 11, n = 40\). Thus we have the new 40-digit amicable numbers:

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* This corrects an error in [2, p. 571, footnote]. W. Borho has asked me to point out here, that his quotation of a correct, private communication of E. J. Lee was incorrect.

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\[ m_1 = 2724918040 \, 3937065577 \, 8575224081 \, 9405848576 = 2^{40}pq, \]
\[ m_2 = 2724918040 \, 3961848563 \, 0625803878 \, 7235905536 = 2^{40}r, \]

with

\[ p = 2^{293} \cdot 683 - 1 = 110 \, 0048498687 \quad (Q = -1), \]
\[ q = 2^{403} \cdot 683 - 1 = 225289 \, 9325313023 \quad (Q = -13), \]
\[ r = 2^{893} \cdot 683^2 - 1 = 24782985 \, 2050580016 \, 6853312511 \quad (Q = -4) \]

and

\[ m_1/m_2 \approx 1 - 2^{-40}. \]

3. Definition. A Thabit-ibn-Kurrah-rule or Thabit-rule

\[ T(b_1, b_2, p, c_1X - 1, c_2X - 1), \]

with given natural numbers \( b_1, b_2, \) a prime \( p, \) and linear polynomials

\[ c_1X - 1, \quad c_2X - 1 \in \mathbb{Z}[X] \]

is a statement of the form:

\[ p^n b_1(c_1p^n - 1) \quad \text{and} \quad p^n b_2(c_2p^n - 1) \]

are amicable numbers, if \( q_i = c_ip^n - 1 \)

is prime and prime to \( b_i \) for \( i = 1, 2 \) \((n = 1, 2, \ldots).\)

For a more general definition see [2].

Walter Borho [2] presents a list of fifteen Thabit-rules, which are constructed from those amicable numbers of the form \( au, \) as \((a, us) = 1, \) \( s \) prime, for which \( p = u + s + 1 \) is prime. Table 1 presents another seven Thabit-rules, constructed in the same way; this completes the list of Thabit-rules which can be constructed from the (at least) 67 published ([8], [9], [10], [2]) amicable pairs of the form \( au, \) as with \((a, us) = 1, \) \( s \) prime.

### Table 1

<table>
<thead>
<tr>
<th>No.</th>
<th>( a )</th>
<th>( u )</th>
<th>( \sigma(u) )</th>
<th>( p )</th>
<th>obtained from pair no.</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i)</td>
<td>( 3^7 \cdot 13 \cdot 19 \cdot 29 )</td>
<td>41 \cdot 173 = 7093</td>
<td>7308</td>
<td>14401</td>
<td>(33) of [3]</td>
</tr>
<tr>
<td>(ii)</td>
<td>( 3^5 \cdot 11^7 )</td>
<td>709 \cdot 2129 = 1509461</td>
<td>1512300</td>
<td>3021761</td>
<td>(31) of [3]</td>
</tr>
<tr>
<td>(iii)</td>
<td>( 3^7 \cdot 11 \cdot 19 \cdot 43 \cdot 89 )</td>
<td>293 \cdot 22961 = 6727573</td>
<td>6750828</td>
<td>13478401</td>
<td>(8) of [5] top of p. 168</td>
</tr>
<tr>
<td>(iv)</td>
<td>( 2^331 )</td>
<td>17 \cdot 107 \cdot 4339 = 7892641</td>
<td>8436960</td>
<td>16329601</td>
<td>(34) of [3]</td>
</tr>
<tr>
<td>(v)</td>
<td>( 2^8 )</td>
<td>257 \cdot 33023 = 8486911</td>
<td>8520192</td>
<td>17007103</td>
<td>(17) of [3]</td>
</tr>
<tr>
<td>(vi)</td>
<td>( 2^319 \cdot 137 )</td>
<td>83 \cdot 218651 = 18148033</td>
<td>18366768</td>
<td>36514801</td>
<td>(2) of [7]</td>
</tr>
<tr>
<td>(vii)</td>
<td>( 2^263 )</td>
<td>4271 \cdot 280883 = 1199651293</td>
<td>1199936448</td>
<td>2399387741</td>
<td>(18) of [3]</td>
</tr>
</tbody>
</table>

In the fifteen Thabit-rules of Borho, and the seven, given here, the numbers \( q_1 = (u + 1)p^n - 1 \) and \( q_2 = (u + 1)\sigma(u)p^n - 1 \) were tested for primality, for all values of \( n \geq 1 \) such that \( q_2 < 10^{120}. \) Both \( q_1 \) and \( q_2 \) appeared to be prime in only three cases; these cases, together with those of Borho and Lee (see [2]) are listed in
Table 2. Table 2 also mentions the discoverers of the amicable pairs from which the Thabit-rules were obtained.

<table>
<thead>
<tr>
<th>Thabit-rule</th>
<th>obtained from an amicable pair discovered by</th>
<th>value of n for which both q₁ and q₂ are primes</th>
<th>discovered by</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 of [2]</td>
<td>Pythagoras (?)</td>
<td>2</td>
<td>Borho</td>
</tr>
<tr>
<td>6 of [2]</td>
<td>Euler</td>
<td>1</td>
<td>Lee</td>
</tr>
<tr>
<td>6 of [2]</td>
<td>Euler</td>
<td>19</td>
<td>te Riele</td>
</tr>
<tr>
<td>(i) of Table 1</td>
<td>Escott</td>
<td>8</td>
<td>te Riele</td>
</tr>
<tr>
<td>(ii) of Table 1</td>
<td>Escott</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

Next follow the details of the three new amicable pairs. Thabit-rule 6 of [2], n = 19, yields the 152-digit amicable numbers:

\[ m₁ = 86 \times 2593766501 \times 4359638769 \times 0953818787 \times 1666597148 \times 4088835777 \]

\[ = 3^{45} \cdot 11 \cdot 5281^{19} \cdot 89 \cdot q₁, \]

\[ m₂ = 90 \times 2364653062 \times 3313066515 \times 5201592687 \times 0786444130 \times 4548569003 \]

\[ = 3^{45} \cdot 11 \cdot 5281^{19 \cdot q₂}, \]

with

\[ q₁ = 2 \cdot 1291 \cdot 5281^{19} - 1 = 13917 \times 5701888775 \times 9763088555 \times 3289918626 \]

\[ = 327^{213} \cdot 19 \cdot 29 \cdot 14401^{8} \cdot 41 \cdot 173 \cdot q₁, \]

\[ q₂ = 2^{33} \cdot 5^{3} \cdot 1291 \cdot 5281^{19} - 1 = 37577439 \times 5099695136 \times 0339099388 \times 2780292340 \]

\[ = 327^{213} \cdot 19 \cdot 29 \cdot 14401^{8} \cdot q₂, \]

and \( m₁/m₂ = .955926. \)

Thabit-rule (i), n = 8, yields the 81-digit amicable numbers:

\[ m₁ = 5 \times 4392258330 \times 0492702317 \times 4526035140 \times 9264518101 \times 4270450011 \]

\[ = 3^{72} \cdot 13 \cdot 19 \cdot 29 \cdot 14401^{8} \cdot 41 \cdot 173 \cdot q₁, \]

\[ m₂ = 5 \times 6040973336 \times 5289816514 \times 3019352151 \times 4014539485 \times 6000068942 \]

\[ = 3^{72} \cdot 13 \cdot 19 \cdot 29 \cdot 14401^{8} \cdot q₂, \]

with

\[ q₁ = 2 \cdot 3547 \cdot 14401^{8} - 1 = 13122977 \times 5913527520 \times 5302674846 \times 4924755893 \]

\( (Q = -15), \)
\[ q_2 = 2^{3^2} \cdot 29 \cdot 3547 \cdot 14401^8 - 1 \]
\[ = 95902720237 \cdot 6059120035 \cdot 1947778167 \cdot 0116073351 \quad (Q = -24) \]

and \( m_1/m_2 = .970580 \).

Thabit-rule (ii), \( n = 1 \), yields the 32-digit amicable numbers:

\[ m_1 = 72 \cdot 387414476 \cdot 8207595520 \cdot 9400624355 = 3^4 \cdot 5 \cdot 11^2 \cdot 71 \cdot 3021761 \cdot 709 \cdot 2129 \cdot q_1, \]
\[ m_2 = 72 \cdot 5235579669 \cdot 5217952865 \cdot 9056738845 = 3^4 \cdot 5 \cdot 11^2 \cdot 71 \cdot 3021761 \cdot q_2, \]

with

\[ q_1 = 2 \cdot 3^2 \cdot 27953 \cdot 3021761 - 1 = 456 \cdot 1233402581 \quad (Q = -3), \]
\[ q_2 = 2^3 \cdot 5^2 \cdot 71 \cdot 27953 \cdot 3021761 - 1 = 689795327 \cdot 4724758599 \quad (Q = -3) \]

and \( m_1/m_2 = .998123 \).

Remark. The two previous examples, of size 81D and 152D, offer contrary evidence to the conjecture [1] that if there exists an infinity of amicable pairs \( (m_1, m_2) \) with \( m_1 < m_2 \), then \( \lim_{m_1 \to \infty} m_1/m_2 = 1 \).

4. Table 2 of [2] lists five more Thabit-rules, which differ slightly from the Thabit-rules mentioned above in Section 3. The numbers \( q_1 \) and \( q_2 \) occurring in these five Thabit-rules were also tested for primality, for all \( n \leq 1 \) with \( q_2 < 10^{120} \). The results were negative in the sense that no pairs \( (q_1, q_2) \) were found with both \( q_1 \) and \( q_2 \) prime.

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