Four Large Amicable Pairs

By H. J. J. te Riele

Abstract. This note gives a report of systematic computer tests of Euler's rule and several Thabit-ibn-Kurrah-rules, in search of large amicable pairs. The tests have yielded four amicable pairs, which are much larger than the largest amicable pair thus far known.

1. The pair of 25-digit numbers

\[(45222 6553454520 8537974785, 45398 0132623392 8286140415)\]

has been the largest known amicable pair since 1946 ([8], [10]). This note gives four new amicable pairs with 32-, 40-, 81-, and 152-digit numbers, as a result of systematic computer tests by Euler's rule (Section 2) and several Thabit-ibn-Kurrah-rules (Sections 3 and 4).

In this research, primality of very large numbers \( N \) had to be established, where \( N + 1 \) can be easily factorized; this was done by use of the following:

**Theorem (Lucas-Lehmer [11, p. 442]).** Let \( P \) and \( Q \) be relatively prime integers and let \( U_0 = 0, U_1 = 1, U_{i+1} = PU_i - QU_{i-1} \) for \( i \geq 1 \). If \( N \) is a natural number, relatively prime to \( 2P - 8Q \), and if \( U_{N+1} \mod N = 0 \), while \( U_{(N+1)/p} \mod N \neq 0 \) for each prime \( p \) dividing \( N + 1 \), then \( N \) is prime.

It is convenient to choose \( P = 1 \), while \( Q \) has to be chosen such that \( D = P - AQ \). In the sequel, the indication "\((Q = A)" after a number means that primality of that number was established by use of this Lucas-Lehmer theorem, with \( Q = A \).

The computations were carried out on the Electrologica-X8 computer of the Mathematical Centre; the value of \( U_i \mod N \) was computed in \( O(\log i) \) steps by use of the binary method (see [6, p. 360 (Exercise 15) and p. 421 (Exercise 26)]).

2. Euler's rule [4] for amicable numbers is given by: \( 2^npq \) and \( 2^nr \) are amicable numbers, if the three integers \( p = 2^x - m - 1, q = 2^y - 1 \) and \( r = 2^z - m - 1 \) are primes, with \( f = 2^x + 1 \) and \( n > m \geq 1 \). For \( m = 1 \), this rule is due to Thabit ibn Kurrah and yields amicable numbers for \( n = 2, 4, 7 \), but for no other value \( n \leq 1000 \) (see [12, p. 874]).* Only one more solution of Euler's rule was known thus far, viz., \( m = 7, n = 8 \) (Legendre, Chebyshev).

A systematic computer search for triples \((p, q, r)\) such that both \( p, q \) and \( r \) are primes was carried out for all values of \( n, m \) with \( n > m > 1 \) and \( r < 10^{132} \); this search yielded just one new solution, viz., \( m = 11, n = 40 \). Thus we have the new 40-digit amicable numbers:

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* This corrects an error in [2, p. 571, footnote]. W. Borho has asked me to point out here, that his quotation of a correct, private communication of E. J. Lee was incorrect.

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3. Definition. A Thabit-ibn-Kurrah-rule or Thabit-rule
\[ T(b_1, b_2, p, c_1X - 1, c_2X - 1), \]
with given natural numbers \( b_1, b_2, \) a prime \( p, \) and linear polynomials
\[ c_1X - 1, c_2X - 1 \in \mathbb{Z}[X] \]
is a statement of the form:
\[ p^n b_1 (c_1 p^n - 1) \text{ and } p^n b_2 (c_2 p^n - 1) \text{ are amicable numbers, if } q_i = c_i p^n - 1 \]
is prime and prime to \( b_i, \) for \( i = 1, 2 \) (\( n = 1, 2, \cdots \)).

For a more general definition see [2].

Walter Borho [2] presents a list of fifteen Thabit-rules, which are constructed from those amicable numbers of the form \( au, \) as (with \( (a, us) = 1, s \) prime), for which \( p = u + s + 1 \) is prime. Table 1 presents another seven Thabit-rules, constructed in the same way; this completes the list of Thabit-rules which can be constructed from the (at least) 67 published ([8], [9], [10], [2]) amicable pairs of the form \( au, \) as with \( (a, us) = 1, s \) prime.

<table>
<thead>
<tr>
<th>No.</th>
<th>( a )</th>
<th>( u )</th>
<th>( \sigma(u) )</th>
<th>( p )</th>
<th>obtained from pair no.</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i)</td>
<td>( 3^7 \times 13 \times 19 \times 29 )</td>
<td>( 41 \times 173 \times 7093 )</td>
<td>7308</td>
<td>14401</td>
<td>(33) of [3]</td>
</tr>
<tr>
<td>(ii)</td>
<td>( 3^5 \times 11^2 \times 71 )</td>
<td>( 709 \times 2129 \times 1509461 )</td>
<td>1512300</td>
<td>3021761</td>
<td>(31) of [3]</td>
</tr>
<tr>
<td>(iii)</td>
<td>( 3^7 \times 11 \times 19 \times 43 \times 89 )</td>
<td>( 293 \times 22961 \times 6727573 )</td>
<td>6750828</td>
<td>13478401</td>
<td>(8) of [5] top of p. 168</td>
</tr>
<tr>
<td>(iv)</td>
<td>( 2^{31} )</td>
<td>( 17 \times 107 \times 4339 \times 7892641 )</td>
<td>8436960</td>
<td>16329601</td>
<td>(34) of [3]</td>
</tr>
<tr>
<td>(v)</td>
<td>( 2^8 )</td>
<td>( 257 \times 33023 \times 8486911 )</td>
<td>8520192</td>
<td>17007103</td>
<td>(17) of [3]</td>
</tr>
<tr>
<td>(vi)</td>
<td>( 2^{19} \times 137 )</td>
<td>( 83 \times 218651 \times 18148033 )</td>
<td>18366768</td>
<td>36514801</td>
<td>(2) of [7]</td>
</tr>
<tr>
<td>(vii)</td>
<td>( 2^{263} )</td>
<td>( 4271 \times 280883 \times 1199651293 )</td>
<td>1199936448</td>
<td>2399387741</td>
<td>(18) of [3]</td>
</tr>
</tbody>
</table>

In the fifteen Thabit-rules of Borho, and the seven, given here, the numbers \( q_1 = (u + 1) p^n - 1 \) and \( q_2 = (u + 1) \sigma(u) p^n - 1 \) were tested for primality, for all values of \( n \geq 1 \) such that \( q_2 < 10^{120} \). Both \( q_1 \) and \( q_2 \) appeared to be prime in only three cases; these cases, together with those of Borho and Lee (see [2]) are listed in
FOUR LARGE AMICABLE PAIRS

Table 2. Table 2 also mentions the discoverers of the amicable pairs from which the Thabit-rules were obtained.

<table>
<thead>
<tr>
<th>Thabit-rule</th>
<th>obtained from an amicable pair discovered by</th>
<th>value of $n$ for which both $q_1$ and $q_2$ are primes</th>
<th>discovered by</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 of [2]</td>
<td>Pythagoras (?)</td>
<td>2</td>
<td>Borho</td>
</tr>
<tr>
<td>6 of [2]</td>
<td>Euler</td>
<td>1</td>
<td>Lee</td>
</tr>
<tr>
<td>6 of [2]</td>
<td>Euler</td>
<td>19</td>
<td>te Riele</td>
</tr>
<tr>
<td>(i) of Table 1</td>
<td>Escott</td>
<td>8</td>
<td>te Riele</td>
</tr>
<tr>
<td>(ii) of Table 1</td>
<td>Escott</td>
<td>1</td>
<td>te Riele</td>
</tr>
</tbody>
</table>

Next follow the details of the three new amicable pairs. Thabit-rule 6 of [2], $n = 19$, yields the 152-digit amicable numbers:

$$m_1 = 86 \cdot 2593766501 \cdot 4359638769 \cdot 0953818787 \cdot 1666597148 \cdot 4088835777$$

$$= 3^{45} \cdot 11 \cdot 5281^{19} \cdot 92^{89} \cdot q_1,$$

$$m_2 = 90 \cdot 2364653062 \cdot 3313066515 \cdot 5201592687 \cdot 0786444130 \cdot 4548569003$$

$$= 3^{45} \cdot 11 \cdot 5281^{19} \cdot q_2,$$

with

$$q_1 = 2 \cdot 1291 \cdot 5281^{19} - 1 = 13917 \cdot 5701888775 \cdot 9763088555 \cdot 3289918626$$

$$= 3^{45} \cdot 11 \cdot 5281^{19} q_1,$$

$$q_2 = 2 \cdot 3^5 \cdot 1291 \cdot 5281^{19} - 1 = 37577439 \cdot 5099695136 \cdot 0339099388 \cdot 2780292340$$

$$= 3^{45} \cdot 11 \cdot 5281^{19} q_2,$$

and $m_1/m_2 = .955926$.

Thabit-rule (i), $n = 8$, yields the 81-digit amicable numbers:

$$m_1 = 5 \cdot 4392258330 \cdot 0492702317 \cdot 4526035140 \cdot 9264518101 \cdot 4270450011$$

$$= 3^{72} \cdot 13 \cdot 19 \cdot 29 \cdot 14401^8 \cdot 41 \cdot 173 \cdot q_1,$$

$$m_2 = 5 \cdot 604097336 \cdot 5289816514 \cdot 3019352151 \cdot 4014539485 \cdot 6000068942$$

$$= 3^{72} \cdot 13 \cdot 19 \cdot 29 \cdot 14401^8 q_2,$$

with

$$q_1 = 2 \cdot 3547 \cdot 14401^8 - 1 = 13122977 \cdot 5913527520 \cdot 5302674846 \cdot 4924755893$$

$$= 3^{72} \cdot 13 \cdot 19 \cdot 29 \cdot 14401^8 q_2,$$

$$q_2 = 2 \cdot 3547 \cdot 14401^8 - 1 = 741 \cdot 5913527520 \cdot 5302674846 \cdot 4924755893$$

$$= 3^{72} \cdot 13 \cdot 19 \cdot 29 \cdot 14401^8 q_2,$$

with

$$Q = -15.$$
\[ q_2 = 2^{3^3} \cdot 29 \cdot 3547 \cdot 14401^8 - 1 = 9590272023 \quad 6059120035 \quad 1947778167 \quad 0116073351 \quad (Q = -24) \]
and \( m_1/m_2 = 0.970580 \).

Thabit-rule (ii), \( n = 1 \), yields the 32-digit amicable numbers:
\[ m_1 = 372 \quad 411111414476 \quad 2807595520 \quad 9400624355 = 3^4 \cdot 11^2 \cdot 71 \cdot 3021761 \cdot 709 \cdot 2129 \cdot q_1, \]
\[ m_2 = 72 \quad 5235579669 \quad 5217952865 \quad 9056738845 = 3^5 \cdot 11^2 \cdot 71 \cdot 3021761 \cdot q_2, \]
with
\[ q_1 = 2^3 \cdot 27953 \cdot 3021761 - 1 = 456 \quad 1233402581 \quad (Q = -3), \]
\[ q_2 = 2^3 \cdot 5^2 \cdot 71 \cdot 27953 \cdot 3021761 - 1 = 689795327 \quad 4724758599 \quad (Q = -3) \]
and \( m_1/m_2 = 0.998123 \).

Remark. The two previous examples, of size 81D and 152D, offer contrary evidence to the conjecture [1] that if there exists an infinity of amicable pairs \((m_1, m_2)\) with \( m_1 < m_2 \), then \( \lim_{m \to \infty} m_1/m_2 = 1 \).

4. Table 2 of [2] lists five more Thabit-rules, which differ slightly from the Thabit-rules mentioned above in Section 3. The numbers \( q_1 \) and \( q_2 \) occurring in these five Thabit-rules were also tested for primality, for all \( n \geq 1 \) with \( q_2 < 10^{120} \). The results were negative in the sense that no pairs \((q_1, q_2)\) were found with both \( q_1 \) and \( q_2 \) prime.

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