Inequalities for Modified Bessel Functions

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Abstract. A sequence of sharp versions of the inequality

$$I_{v+1}(x) < I_v(x),$$

where $v > -\frac{1}{2}$ and $x > 0$, was established by Soni [8] in 1965. Results that are stronger than (1) for $v \geq 0$ have been given by Jones [3], Cochran [2], and Reudink [7]. Thus, Jones proved that

$$I_{v}(x) < I_{v+1}(x),$$

for $v > 0$ and $x > 0$, while Cochran established the inequality $\partial I_v(x)/\partial v < 0$ for $v \geq 0$ and $x > 0$. Reudink, apparently unaware of the work of the previous authors, proved in a different way that $\partial I_v(x)/\partial v < 0$ for $v > 0$ and $x > 0$.

We observe first that (1) holds for $v \geq -\frac{1}{2}$. Indeed, with $x > 0$, we have

$$I_{v}(x) - I_{v+1}(x) = \frac{i^2}{\pi} \frac{1}{x^v} e^{-x} > 0.$$

In the present note, we prove two propositions. The first one contains a rather modest but easily proved result that strengthens (1) for $v > 0$. The second proposition gives a sequence of progressively sharper lower bounds of $I_v(x)$ that converge monotonically to $I_v(x)$.

**Proposition 1.** Let $v \geq -1$ and $x > 0$. Then

$$I_{v+1}(x) < I_v(x).$$

*Proof.* The series representation for $I_v(x)$ is

$$I_v(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{2k}}{k! (k + v + 1)}.$$

Setting $k + 1 = j$ gives

$$I_v(x) = \sum_{j=1}^{\infty} \frac{(x/2)^{2j}}{j! (j + v + 1)} j(j + v).$$

The average of these two expressions is

$$I_v(x) = \frac{1}{2} \left( \sum_{k=0}^{\infty} \frac{(x/2)^{2k}}{k! (k + v + 1)} \right) \left[ \left( \frac{x}{2} \right)^2 + k(k + v) \right].$$
Replacing \( \nu \) by \( \nu + 1 \) in (4), multiplying by \((1 + \nu/x)\), and subtracting the resulting expression from (5) gives

\[
I_\nu(x) - \left(1 + \frac{\nu}{x}\right)I_{\nu+1}(x) = \frac{1}{2} \left(\frac{x}{2}\right)^{\nu-2} \sum_{k=0}^{\infty} \frac{(\nu/2)^{2k}}{k! \Gamma(k + \nu + 1)} \left(\frac{x}{2} - k\right)^2 > 0,
\]

which proves (3).

This result was established by discarding an infinite series of nonnegative terms. A sharp version of (3) results from retaining a finite number of these terms.

As a preparation for Proposition 2, we define two sequences of functions \( \{G_{\nu,k}\} \) and \( \{H_{\nu,k}\}, k = 0, 1, 2, \ldots, \) by*

\[
G_{\nu,k}(x) = \sum_{i=1}^{k} (-1)^{i+1} \binom{k}{i} \frac{2(\nu + 1)_{i-1} (\nu + j)}{(2\nu + k + 1)_i} I_{\nu+i}(x)
\]

and

\[
H_{\nu,k}(x) = G_{\nu,k}(x) + \frac{1}{\Gamma(\nu + 1)} \left(\frac{x}{2}\right)^\nu e^{-x},
\]

where \( \nu > -\frac{1}{2} \) and \( x > 0 \).

We note that the inequality \( G_{\nu,k}(x) < H_{\nu,k}(x) \) follows from these definitions.

**Proposition 2.** Let \( \nu > -\frac{1}{2} \) and \( x > 0 \). Then

(i) \( 0 < H_{\nu,k}(x) < H_{\nu+1,k}(x) < I_{\nu}(x), \quad k \geq 0, \)

(ii) \( H_{\nu,k}(x) \sim I_{\nu}(x), \quad x \to 0, \quad k \geq 0, \)

(iii) \( I_{\nu}(x) - H_{\nu,k}(x) \sim \frac{(2\nu + 1)_k}{(2x)^k} I_{\nu}(x), \quad x \to \infty, \quad k \geq 0, \)

and

(iv) \( \lim_{k \to \infty} H_{\nu,k}(x) = I_{\nu}(x). \)

**Proof.** Our proof is based on an expansion of the confluent hypergeometric function in terms of modified Bessel functions. The following expression follows from Luke [4, p. 48]:

\[
_1F_1(a; c; z) = \Gamma(a + \frac{1}{2}) x^{a/2} \left(\frac{d}{z}\right)^{a-1/2}
\]

\[
\cdot \left[ I_{a-1/2}(\frac{z}{2}) + \sum_{j=1}^{\infty} \frac{2(j + a - \frac{1}{2})(2a)_{j-1}(2a - c)_j}{j! (c)_j} I_{j+a-1/2}(\frac{z}{2}) \right].
\]

By letting \( k \geq 0 \) be an integer and putting \( a = \nu + \frac{1}{2}, c = 2\nu + k + 1, \) and \( z = 2x, \) we find from this expression and (6) that

\[
I_{\nu}(x) - G_{\nu,k}(x) = \frac{1}{\Gamma(\nu + 1)} \left(\frac{x}{2}\right)^\nu e^{-x} _1F_1(\nu + \frac{1}{2}; 2\nu + k + 1; 2x).
\]

But \( _1F_1(a; c; z) > 1 \) for \( a > 0, c > 0, z > 0. \) We conclude therefore from (8) and (7) that \( I_{\nu}(x) > H_{\nu,k}(x). \)

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* We adopt the convention \( \sum_{k=0}^{n} \alpha_k = 0 \) for \( n < m \) and the notation \( (\alpha)_n = \alpha(\alpha + 1) \cdots (\alpha + n - 1), n \geq 1, (\alpha)_0 = 1. \)
From the contiguous recurrence relations for the confluent hypergeometric function, we find that

\[ {}_1F_1(a; c; z) - {}_1F_1(a; c + 1; z) = \frac{az}{c(c + 1)} {}_1F_1(a + 1; c + 2; z). \]

From this recurrence relation and (8), we get

\[ G_{r,k+1}(x) - G_{r,k}(x) = \frac{2(2\nu + 1)}{\Gamma(\nu + 1)(2\nu + k + 1)} \left( \frac{x}{2} \right)^{\nu + 1} e^{-x} {}_1F_1(\nu + \frac{3}{2}; 2\nu + k + 3; 2x). \]

Since the right-hand side of this equality is positive, we conclude that

\[ H_{r,k+1}(x) - H_{r,k}(x) = G_{r,k+1}(x) - G_{r,k}(x) > 0. \]

This establishes (i) since \( H_{r,0}(x) > 0 \).

By using the first two terms in the series expansions of \( I_{\nu}(x), e^{-x} \), and \( {}_1F_1(\nu + \frac{3}{2}; 2\nu + k + 1; 2x) \), we find from (8) that the asymptotic behavior of \( G_{r,k}(x) \) as \( x \to 0 \) is

\[ G_{r,k}(x) \sim \frac{2k}{(2\nu + k + 1)\Gamma(\nu + 1)} \left( \frac{x}{2} \right)^{\nu + 1}, \quad k \geq 0. \]

From (7), we therefore get

\[ H_{r,k}(x) \sim \frac{1}{\Gamma(\nu + 1)} \left( \frac{x}{2} \right)^{\nu} \]

as \( x \to 0 \) for \( k \geq 0 \). This establishes (ii).

We next apply the asymptotic expansion of \( {}_1F_1(\nu + \frac{3}{2}; 2\nu + k + 1; 2x) \) as \( x \to \infty \) to (8) and use the duplication formula for the \( \Gamma \)-function to establish the asymptotic relation

\[ I_{\nu}(x) - G_{r,k}(x) \sim \frac{(2\nu + 1)k}{(2x)^{\nu + 1}} e^{-x} \]

as \( x \to \infty \) and \( k \geq 0 \). Statement (iii) now follows from (7) and the asymptotic expansion for \( I_{\nu}(x) \) as \( x \to \infty \).

Statement (iv) follows via relations (8) and (7) from the observation that \( \lim_{x \to 0} {}_1F_1(a; c; z) = 1. \)

We proceed to compare the inequality \( H_{r,k}(x) < I_{\nu}(x) \) with (1), (2), and (3).

From the definition of \( G_{r,k} \) in (6), we find \( G_{r,k}(x) = I_{\nu+k}(x) \). Hence, the inequality \( H_{r,k}(x) < I_{\nu}(x) \) is sharper than (1) for all \( k \geq 1 \).

Let \( \mu > \nu \geq 0 \) be fixed. We find then from (ii) that the inequality \( H_{r,k}(x) < I_{\nu}(x) \) is sharper than (2) for all \( k \geq 0 \), provided \( x \) is sufficiently close to 0. The asymptotic behavior as \( x \to \infty \) of \( I_{\nu}(x) - I_{\nu+k}(x) \) is

\[ I_{\nu}(x) - I_{\nu+k}(x) \sim \frac{\mu^2 - \nu^2}{2x} I_{\nu}(x). \]

A comparison with (iii) shows that \( H_{r,k}(x) < I_{\nu}(x) \) is a sharper inequality than (2) for \( k \geq 2 \) and all sufficiently large values of \( x \).

In order to effect a comparison with (3), we put \( k = 2 \) in (6) to get
The recurrence relation

\[ I_{\nu+2}(x) = I_{\nu}(x) - \frac{2(\nu + 1)}{x} I_{\nu+1}(x) \]

then gives

\[ I_{\nu}(x) - G_{\nu,2}(x) = \frac{2(\nu + 1)}{\nu + \frac{3}{2}} \left[ I_{\nu}(x) - \left( 1 + \frac{\nu + \frac{3}{2}}{x} \right) I_{\nu+1}(x) \right]. \]

Hence, the inequality \( H_{\nu,2}(x) < I_{\nu}(x) \) is stronger than (3) for \( k \geq 2 \) and \( \nu > -\frac{1}{2} \).

The inequalities discussed here are all in the form of lower bounds of \( I_{\nu}(x) \). An upper bound of \( I_{\nu}(x) \) is derived as follows. Replace \( \nu \) by \( \nu + 1 \) in (10), eliminate \( I_{\nu+2}(x) \) from the bracket in the right-hand side of (10) by using (9), and make use of the positivity of the bracket. It follows that

\[ I_{\nu}(x) < \frac{1 + 2(\nu + 1)/x + 2(\nu + 1)(\nu + \frac{3}{2})/x^2}{1 + (\nu + \frac{3}{2})/x} I_{\nu+1}(x) \]

for \( x > 0 \) and \( \nu > -\frac{3}{2} \). Sharp versions of (11) are derived by making use of the inequalities \( I_{\nu}(x) > H_{\nu,2}(x), k \geq 2 \) or \( I_{\nu}(x) > G_{\nu,2}(x), k \geq 3 \). The general form for these upper bounds has not been found.

Luke [5] and Prohorov [6] have given inequalities for modified Bessel functions. These inequalities are weaker than those discussed here but have the virtue that the bounds for \( I_{\nu}(x) \) are easily evaluated numerically.

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[References omitted for brevity]