On Semicardinal Quadrature Formulae*

By I. J. Schoenberg and S. D. Silliman

Abstract. The present paper concerns the semicardinal quadrature formulae introduced in Part III of the reference [3]. These were the limiting forms of Sard's best quadrature formulae as the number of nodes increases indefinitely. Here we give a new derivation and characterization of these formulae. This derivation uses appropriate generating functions and also allows us to compute the coefficients very accurately.

Introduction. The present paper is a slightly shortened version of the MRC report [5]. Let \( m \) be a natural number and let

\[ \mathcal{S}_{2m-1} = \{ S(x) \} \]

denote the class of functions \( S(x) \) satisfying the three conditions:

\[ S(x) \in C^{2m-2}(\mathbb{R}), \]
\[ S(x) \in \mathcal{P}_{2m-1} \] in each of the intervals \((0, 1), (1, 2), \ldots,\)
\[ S(x) \in \mathcal{P}_{m-1} \] in the interval \((-\infty, 0)\).

These functions are the so-called natural semicardinal splines of degree \( 2m - 1 \).

It was shown in [3, Lemma 5, Section 9] that if

\[ S(x) \in \mathcal{S}_{2m-1} \cap L_1(\mathbb{R}^+), \]

then

\[ \sum_{\nu=0}^{\infty} |S(\nu)| < \infty. \]

It follows that, if \( B_n \) is a sequence of constants such that

\[ B_n = O(1) \quad \text{as} \quad \nu \to \infty, \]

then the functional

\[ RS = \int_0^\infty S(x) \, dx - \sum_{\nu=0}^{\infty} B_n S(\nu) \]

is well defined for every \( S(x) \) satisfying (5).

In the same paper [3, Theorem 6, Section 10], the following theorem was established.

**Theorem 1.** We consider a quadrature formula

\[ \int_0^\infty f(x) \, dx = \sum_{\nu=0}^{\infty} B_n f(\nu) + R \]

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with perfectly arbitrary constant coefficients $B$, subject only to the condition (7). Among these formulae, there is exactly one with the property that

$$Rf = 0 \quad \text{whenever} \quad f(x) \in S_{2m-1} \cap L_1(\mathbb{R}^+).$$

We denote this unique formula by

$$\int_0^\infty f(x) \, dx = \sum_{\nu=0}^{\infty} H_{\nu}^{(m)} f(\nu) + Rf$$

and call it the semicardinal quadrature formula of order $m$.

For the derivation of (11) by integrating the semicardinal interpolation formula, see [3, Section 10], wherein its connection with some conjectures due to L. F. Meyers and A. Sard concerning best quadrature formulae is explained (see also [4, Lecture 8]).

The purpose of the present note is the accurate computation of the values of the coefficients $H_{\nu}^{(m)}$ for $m = 2, 3, \ldots, 7$. The tables of Sections 7 and 8 are based on computations beautifully performed by Mrs. Julia Gray, of the Computing Staff of the Mathematics Research Center, on the CDC 3600. They were done in double precision and all decimals listed should be correct, as we had anywhere from 17 to 24 correct decimals throughout. The zeros of the Euler-Frobenius polynomials of Section 7 were found by the algorithm due to D. H. Lehmer. It seems of some interest to observe that

$$H_4^{(7)} < 0.$$ 

We also give a new proof of Theorem 1 which is simpler than the proof presented in [3, Section 10] where the main emphasis was in establishing the harder Meyers-Sard conjectures.

We conclude this Introduction by mentioning two further remarkable semicardinal formulae: The first is the Euler-Maclaurin formula

$$\int_0^\infty f(x) \, dx = \frac{1}{2} f(0) + f(1) + f(2) + \cdots + \sum_{r=1}^{m-1} \frac{B_{2r}}{(2r)!} f^{(2r-1)}(0) + Rf.$$ 

The second is the so-called complete semicardinal formula

$$\int_0^\infty f(x) \, dx = \sum_{\nu=0}^{\infty} \tilde{H}_{\nu}^{(m)} f(\nu) + \sum_{i=1}^{m-1} A_i^{(m)} f^{(i)}(0) + Rf.$$ 

Both formulae are uniquely defined among quadrature formulae of their type (i.e., when all their terms are provided with arbitrary coefficients subject only to the condition that the coefficients of $f(\nu)$ should form a bounded sequence) by the condition of being exact, hence $Rf = 0$, whenever $f(x)$ is any spline of degree $2m - 1$ in the interval $[0, +\infty)$, with knots at $1, 2, \ldots$, such that $f(x) \in L_1(\mathbb{R}^+)$. Among the formulae (11), (12), and (13), the formula (13) is, as a rule, the most accurate in numerical applications (after an appropriate change of step), while (11) is the least accurate. The computation of the coefficients of the complete formula (13) is the subject of Silliman’s forthcoming paper [6].

The reader will see that the use of the $B$-splines (Section 1) transforms a fairly formidable problem into one that is within easy reach of the Euler-Laplace method of generating functions.
I. The Construction of the Semicardinal Quadrature Formula.

1. B-Splines and Euler-Frobenius Polynomials. Here we collect tools and results that have proved to be indispensable in the study of cardinal splines. Writing \( x_* = \max(0, x) \), the forward B-spline is defined by (see [1, Section 1])

\[
Q_m(x) = \frac{1}{(m - 1)!} \sum_{i=0}^{m} (-1)^i \binom{m}{i} (x-i+1)^{m-1} \quad (x \in \mathbb{R}).
\]

This is a spline function of degree \( m - 1 \), with knots at \( x = 0, 1, \ldots, m \). The symmetry property \( Q_m(x) = Q_m(m - x) \) shows that we may equivalently write it in the form

\[
Q_m(x) = \frac{1}{(m - 1)!} \sum_{i=0}^{m} (-1)^{m-i} \binom{m}{i} (v-x)^{m-1}.
\]

This is a frequency function. More precisely,

\[
Q_m(x) > 0 \quad \text{if} \quad 0 < x < m, \quad Q_m(x) = 0 \quad \text{if} \quad x \leq 0, \text{or} \quad x \geq m.
\]

Euler's generating function

\[
\frac{x-1}{x-e^x} = \sum_{n=0}^{\infty} \frac{\Pi_n(x)}{(x-1)^n} \frac{x^n}{n!}
\]

defines the polynomial \( \Pi_n(x) \) of degree \( n - 1 \), called the Euler-Frobenius polynomial. For proofs of its properties described below in Lemma 1, we refer to [2, Lemma 7].

**Lemma 1.**

(i) \( \Pi_n(x) \) is a reciprocal monic polynomial of degree \( n - 1 \) with integer coefficients satisfying the recurrence relation

\[
\Pi_{n+1}(x) = (1 + nx)\Pi_n(x) + x(1-x)\Pi_n'(x) \quad (\Pi_1(x) = 1).
\]

(ii) The identity

\[
\Pi_n(x)/(1-x)^{n+1} = \sum_{\nu=0}^{\infty} (\nu+1)x^{\nu} \quad (|x| < 1),
\]

holds.

(iii) The zeros \( \lambda_* \) of \( \Pi_n(x) \) are all simple and negative. We label them so that

\[
\lambda_{n-1} < \lambda_{n-2} < \ldots < \lambda_2 < \lambda_1 < 0.
\]

(iv) The identity

\[
\Pi_n(x) = n! \sum_{\nu=0}^{n-1} Q_{n+1}(\nu+1)x^\nu
\]

holds.

The identity (1.8) shows the close relation between B-splines and Euler-Frobenius polynomials. In Section 7, the reader will find the polynomials \( \Pi_{2m-1}(x) \) and their zeros for \( m = 2, 3, 4, 5, 6, \) and 7.

2. A Recurrence Relation. In Sections 2, 3, and 4, we determine the Q.F. (9) satisfying conditions (7) and (10). To begin with, we ignore condition (7) and argue as follows.

We integrate the B-spline (1.2) \( m \) times so as to preserve the vanishing of the
function in \((m, + \infty)\). This condition uniquely defines the integral

\[
\sigma(x) = \sigma_m(x) = \frac{1}{(2m - 1)!} \sum_{\nu=0}^{m} (-1)^\nu \binom{m}{\nu}(x - \nu)^{2m-1}
\]

having the properties

\[
\sigma^{(m)}(x) = \mathcal{Q}_m(x), \quad \sigma(x) = 0 \quad \text{if} \quad x \leq m.
\]

Moreover, since \(\mathcal{Q}_m(x) = 0\) if \(x \leq 0\), we conclude that

\[
\sigma(x) \in S_{2m-1}^+ \cap L_1(\mathbb{R}^+).
\]

Clearly, this property of \(\sigma(x)\) will remain valid if we shift its graph to the right by an integer amount, hence

\[
\sigma(x - n) \in S_{2m-1}^+ \cap L_1(\mathbb{R}^+) \quad \text{for} \quad n = 0, 1, 2, \cdots.
\]

We conclude: The coefficients \(B_\nu\) of an arbitrary Q.F. \((9), (7)\), that enjoys the property \((10)\), must satisfy the relations

\[
\int_0^{n+m} \sigma(x - n) \, dx = \sum_{\nu=0}^{n+m-1} B_\nu \sigma(\nu - n) \quad (n = 0, 1, 2, \cdots).
\]

The series on the right side indeed breaks off as indicated because of the second relation \((2.2)\).

3. The Summation of Certain Power Series. The structure of the relations \((2.5)\) suggests the use of generating functions for the determination of the \(B_\nu\). Indeed, the right side of \((2.5)\) is seen to be equal to the coefficient of \(x^{n+m-1}\) in the product of power series

\[
\left(\sum_{\nu=0}^{\infty} B_\nu x^\nu\right) \left(\sum_{\nu=0}^{\infty} \sigma(m - 1 - \nu)x^\nu\right).
\]

(A) To simplify notations, we define the sequence \((s_\nu; \nu = 0, 1, \cdots)\) by

\[
s_\nu = (-1)^\nu(2m - 1)! \sigma(m - 1 - \nu) = \sum_{r=0}^{m} (-1)^{m+r} \binom{m}{r}(r - m + 1 + \nu)^{2m-1},
\]

or

\[
s_\nu = \sum_{k=0}^{\infty} (-1)^k \binom{m}{k}(\nu + 1 - k)^{2m-1}.
\]

Multiplying by \(x^\nu\) and summing for \(\nu = 0, 1, \cdots\), we obtain

\[
\sum_0^\infty s_\nu x^\nu = \sum_0^\infty (-1)^k \binom{m}{k} \sum_0^\infty (\nu + 1 - k)^{2m-1} x^\nu
\]

\[
= \sum_0^\infty (-1)^k \binom{m}{k} \sum_0^\infty (\nu + 1)^{2m-1} x^{\nu+k}.
\]

Using \((1.6)\), the right side becomes

\[
= \sum_0^\infty (-1)^k \binom{m}{k} x^k \Pi_{2m-1}(x)/(1 - x)^{2m}
\]
and we finally obtain that

\[(3.4) \sum_{0}^{\infty} s_{x^{r}} = \Pi_{2m-1}(x)/(1 - x)^{m}.\]

(B) For the integrand on the left side of (2.5), we find, by (2.1),

\[\sigma(x - n) = \frac{1}{(2m - 1)! \sum_{r=0}^{\infty} (-1)^{r} {m \choose r} (n + x)^{2m-1}},\]

whence

\[(3.5) \int_{0}^{\infty} \sigma(x - n) \, dx = \frac{1}{(2m)! \sum_{r=0}^{\infty} (-1)^{r} {m \choose r} (n + x)^{2m}}.\]

As in (3.2), we introduce the new quantities

\[(3.6) F_{n+m-1} = (-1)^{m}(2m - 1)! \int_{0}^{\infty} \sigma(x - n) \, dx\]

\[= \frac{1}{2m} \sum_{r=0}^{\infty} (-1)^{n+r} {m \choose r} (n + x)^{2m} \quad (n \geq 0),\]

and wish to sum the series

\[(3.7) \sum_{n=0}^{\infty} F_{n+m-1} x^{n+m-1}.\]

From (3.6) we obtain

\[(3.8) 2m \sum_{n=0}^{\infty} F_{n+m-1} x^{n+m-1} = \sum_{r=0}^{\infty} (-1)^{n+r} {m \choose r} \sum_{n=0}^{\infty} (n + x)^{2m} x^{n+m-1},\]

while the inside sum is

\[\sum_{n=0}^{\infty} (n + x)^{2m} x^{n+m-1} = x^{-\nu} \sum_{n=0}^{\infty} (n + x)^{2m} x^{n+r-1} = x^{-\nu} \sum_{r=0}^{\infty} (r + 1)^{2m} x^{r} - x^{-\nu} \sum_{r=0}^{\infty} (r + 1)^{2m} x^{r} \]

\[= x^{-\nu} \Pi_{2m}(x)/(1 - x)^{2m+1} - V_{\nu}(x)\]

by (1.6). Here, \(V_{\nu}(x)\) is an element of \(\pi_{m-2}\). Substituting this into (3.8), we obtain

\[(3.9) \sum_{n=1}^{\infty} F_{n} x^{n} = \frac{1}{2m} \frac{\Pi_{2m}(x)}{(1 - x)^{2m+1}} - \frac{1}{2m} V(x), \quad \text{where } V(x) \in \pi_{m-2}.\]

Evidently, \(V(x)\) is such as to cancel the first \(m - 1\) terms of the power series expansion of the first term on the right side.

The relations (2.5) may now be written as

\[(3.10) F_{n} = \sum_{r=0}^{n} B_{r} s_{n-r}, \quad \text{for } n \geq m - 1.\]

We may here select the first \(m - 1\) terms

\[(3.11) B_{0}, B_{1}, \cdots, B_{m-2} \]
arbitrarily and determine the entire sequence \((B_i)\) recursively by (3.10). Equivalently, we may select the \(m - 1\) quantities \(F_0, F_1, \cdots, F_{m-2}\), arbitrarily and determine \((B_i)\) from the identity

\[
\sum_{0}^{\infty} F_i x^i = \left( \sum_{0}^{\infty} B_i x^i \right) \left( \sum_{0}^{\infty} s_i x^i \right).\]

By (3.4) and (3.9), we have

\[
\sum_{0}^{\infty} s_i x^i = \frac{\Pi_{2m}(x)/(1 - x)^m,}{1 - x} \]

and

\[
\sum_{0}^{\infty} F_i x^i = \frac{1}{2m} \frac{\Pi_{2m}(x)}{1 - x} - \frac{1}{2m} \frac{U(x)}{1 - x}.\]

where \(U(x)\) is an arbitrary element of \(\pi_{m-2}\). Solving (3.12) for \(\sum_{0}^{\infty} B_i x^i\), we obtain the following:

**Theorem 2.** The coefficients \((B_i)\) of the most general functional

\[
Rf = \int_{0}^{\infty} f(x) \, dx - \sum_{0}^{\infty} B_i f(i),
\]

that vanishes for the functions of the sequence

\[
\sigma(x - n) \quad (n = 0, 1, 2, \cdots),
\]

are the expansion coefficients of

\[
R_m(x) = \sum_{0}^{\infty} B_i x^i,
\]

where

\[
R_m(x) = \frac{\Pi_{2m}(x)}{2m(1 - x)\Pi_{2m-1}(x)} - \frac{(1 - x)^m U(x)}{2m\Pi_{2m-1}(x)}.
\]

Here, \(U(x)\) is an arbitrary element of \(\pi_{m-2}\).

4. Determining the Coefficients \(H_{i,m}\). This will be done by requiring the coefficients \((B_i)\) of (3.17) to satisfy (7) or

\[
B_i = O(1) \quad \text{as} \quad \nu \rightarrow \infty.
\]

The order of magnitude of the \(B_i\) for large \(\nu\) is controlled by the location of the poles of the rational function \(R_m(x)\). Let us first transform its expression slightly. From the recurrence relation (1.5), we find that

\[
\Pi_{2m}(x) = (1 + (2m - 1)x)\Pi_{2m-1}(x) + x(1 - x)\Pi'_{2m-1}(x),
\]

and, substituting into (3.18), we obtain that

\[
R_m(x) = \frac{1 + (2m - 1)x}{2m(1 - x)} + \frac{x\Pi'_{2m-1}(x)}{2m\Pi_{2m-1}(x)} - \frac{(1 - x)^m U(x)}{2m\Pi_{2m-1}(x)}.
\]

From (1.7) we know that the \(2m - 2\) zeros \(\lambda\) of \(\Pi_{2m-1}(x)\) are simple and negative. Also, that \(\Pi_{2m-1}(x)\) is a reciprocal polynomial, whence the relations \(\lambda_1\lambda_{2m-2} = \)
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\[ \lambda_2 \lambda_{2m-3} = \cdots = \lambda_{m-1} \lambda_m = 1. \] It follows that these zeros satisfy the inequalities

\[ \lambda_{2m-2} < \cdots < \lambda_m < -1 < \lambda_{m-1} < \cdots < \lambda_1 < 0. \]  

Let

\[ U(x) = a_0 + a_1 x + \cdots + a_{m-2} x^{m-2}. \]

It is now easy to decompose \( R_m(x) \) into partial fractions. Observing that \( R_m(x) \) is regular at \( x = \infty \), we find that

\[
R_m(x) = -\frac{1}{2m} + \frac{1}{2m} (-1)^{m+1} a_{m-2} + \frac{1}{1 - x} \\
+ \frac{1}{2m} \sum_{r=1}^{2m-2} \frac{\lambda_r}{x - \lambda_r} - \frac{1}{2m} \sum_{r=1}^{2m-2} \frac{U(\lambda_r)(1 - \lambda_r)^m}{(x - \lambda_r)\Pi'_{2m-1}(\lambda_r)}.
\]

From (4.3) we see that the poles \( \lambda_1, \cdots, \lambda_{m-1} \) are inside the unit circle, while \( \lambda_m, \cdots, \lambda_{2m-2} \) are outside. Also, \( x = 1 \) is a simple pole, by (4.5). It follows that (4.1) will hold if and only if the polynomial \( U(x) \) can be so chosen that the inside poles \( \lambda_1, \cdots, \lambda_{m-1} \) cancel out, i.e., their residues vanish. An inspection of (4.5) shows this to be the case if and only if \( U(x) \) satisfies the equations

\[ U(\lambda_r) = \lambda_r \Pi_{2m-1}(\lambda_r)(1 - \lambda_r)^{-m} \quad (r = 1, \cdots, m - 1). \]

We see that \( U(x) \) exists uniquely, because (4.6) describes an ordinary Lagrange interpolation problem. This establishes

**Theorem 3.** *There is a unique Q.F.*

\[ \int_0^\infty f(x) \, dx = \sum_{0}^{m} H_r f(x) + Rf \]

*having bounded coefficients and which is exact for the sequence of functions \( \sigma(x - n) \) \( (n = 0, 1, \cdots) \). Its coefficients are given by the expansion*

\[ R_m(x) = \sum_{0}^{m} H_r x^m \quad (|x| < 1). \]

*Here*

\[ R_m(x) = -\frac{1}{2m} + \frac{1}{2m} (-1)^{m+1} a_{m-2} + \frac{1}{1 - x} \\
+ \frac{1}{2m} \sum_{r=1}^{2m-2} \left( \frac{\lambda_r - U(\lambda_r)(1 - \lambda_r)^m}{\Pi'_{2m-1}(\lambda_r)} \right) \frac{1}{x - \lambda_r}, \]

*where \( U(x) = a_{m-2} x^{m-2} + \text{lower degree terms}, is the solution of the interpolation problem (4.6).*

In order to complete a proof of Theorem 1, we are still to show that the remainder functional \( Rf \) of the formula (4.7) satisfies the condition (10) of Theorem 1. For a proof of this, we refer to [5, Section 5].

### 5. Final Computational Details

We return to the rational function \( R_m(x) \), defined by (4.9), that generates the \( H_r(x) \) by (4.8). For even moderately large values of \( m \), the zero \( \lambda_1 \) is small and its reciprocal \( \lambda_{2m-2} \) is correspondingly large (e.g., for \( m = 7 \), we find that \( \lambda_1 = -0.001251 \)). It is therefore important from the computational...
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point of view to express the right side of (4.9) in terms of the zeros \( \lambda_1, \cdots, \lambda_{m-1} \). This is easily done by the following device: We define the new polynomials \( U^* \) and \( \Pi_{2m-1}^* \) by setting

\[
U^*(x) = x^{m-2} U(x^{-1}), \quad \Pi_{2m-1}^*(x) = x^{2m-3}\Pi_{2m-1}'(x^{-1}).
\]

In terms of these polynomials, (4.9) becomes

\[
R_m(x) = C + \frac{1}{1 - x} + \sum_{i=1}^{m-1} C_i \frac{1}{1 - \lambda_i x},
\]

where

\[
C = -\frac{1}{2m} + \frac{1}{2m} (-1)^{m+1} a_{m-2},
\]

\[
C_i = \frac{1}{2m} \left\{ \frac{U^*(\lambda_i)(\lambda_i - 1)^m}{\Pi_{2m-1}^*(\lambda_i)} - 1 \right\} \quad (\nu = 1, \cdots, m - 1).
\]

Expanding the right side of (5.2) in powers of \( x \) and using (4.8), we obtain

**Corollary 1.** The coefficients of the semicardinal Q.F. (11) have the values

\[
\begin{align*}
\gamma_m &= C + 1 + \sum_{i=1}^{m-1} C_i, \\
\gamma_{m-1} &= 1 + \sum_{i=1}^{m-1} C_i \lambda_i \quad (j = 1, 2, \cdots),
\end{align*}
\]

where \( C \) and \( C_i \) are given by (5.3), (5.4).

It is convenient to define

\[
\begin{align*}
h_0^m &= H_0^m - \frac{1}{2}, \\
h_i^m &= H_i^m - 1 \quad (j \geq 1),
\end{align*}
\]

and to write the Q.F. (11) in the form

\[
\int_0^m f(x) \, dx = T + \sum_{i=0}^{m} h_i^m f(\nu) + Rf,
\]

where \( T \) stands for the trapezoidal sum

\[
T = \frac{1}{2} f(0) + \sum_{i=1}^{m} f(\nu).
\]

From (5.5), (5.6) and in view of (5.7), we obtain that

\[
\begin{align*}
h_0^m &= C + \frac{1}{2} + \sum_{i=1}^{m-1} C_i, \\
h_i^m &= \sum_{i=1}^{m-1} C_i \lambda_i \quad (j = 1, 2, \cdots).
\end{align*}
\]

6. The Case \( m = 2 \) of Cubic Splines. We mention this case separately because the results are explicit and also because, for this case, Meyers and Sard established their conjecture. From our formulae (4.4), (4.6), (5.3) to (5.6), we easily find that

\[
\lambda_1 = -2 + \sqrt{3}, \quad a_0 = -\frac{1}{3} \sqrt{3}, \quad C = \frac{-3 + \sqrt{3}}{12}, \quad C_1 = -\frac{1}{2},
\]
and therefore

\[ H^{(2)}_0 = \frac{3 + \sqrt{3}}{12}, \quad H^{(2)}_j = 1 - \frac{j}{12} \lambda_j^2 \quad (j = 1, 2, \ldots). \]

These agree with the values given by Meyers and Sard. For references to the work of Meyers and Sard, see [3].

II. Numerical Results.

7. The Polynomials \( \Pi_{m-1}(x) \) and Their Zeros for \( m = 2, 3, \ldots, 7 \).

\[
\begin{array}{c|c|c|c}
\nu & \lambda_\nu & \lambda_{\nu+1} & \lambda_{\nu+2} \\
1 & -2.6794 & 91924 & 31123 \\
2 & -3.7320 & 08075 & 68877 \\
\end{array}
\]

\[
\begin{array}{c|c|c|c|c}
m = 2: & \Pi_3(x) = x^2 + 4x + 1 & \lambda_\nu & \lambda_{\nu+1} & \lambda_{\nu+2} \\
\nu & \lambda_\nu & \lambda_{\nu+1} & \lambda_{\nu+2} & \lambda_{\nu+3} \\
1 & -0.0430 & 62882 & 03264 & 07743 \\
2 & -0.4305 & 53470 & 99974 & 00000 \\
3 & -0.3224 & 38869 & 40428 & 33333 \\
4 & -0.2320 & 47777 & 56333 & 00000 \\
\end{array}
\]

\[
\begin{array}{c|c|c|c|c}
m = 3: & \Pi_5(x) = x^4 + 26x^3 + 66x^2 + 26x + 1 & \lambda_\nu & \lambda_{\nu+1} & \lambda_{\nu+2} \\
\nu & \lambda_\nu & \lambda_{\nu+1} & \lambda_{\nu+2} & \lambda_{\nu+3} & \lambda_{\nu+4} \\
1 & -0.0043 & 06288 & 32647 & 07743 & 00000 \\
2 & -0.1225 & 51715 & 92326 & 00000 & 00000 \\
3 & -0.3224 & 38869 & 40428 & 33333 & 00000 \\
4 & -0.2320 & 94777 & 63333 & 00000 & 00000 \\
\end{array}
\]

\[
\begin{array}{c|c|c|c|c|c|c}
m = 4: & \Pi_7(x) = x^6 + 120x^5 + 1191x^4 + 2416x^3 + 1191x^2 + 120x + 1 & \lambda_\nu & \lambda_{\nu+1} & \lambda_{\nu+2} & \lambda_{\nu+3} & \lambda_{\nu+4} \\
\nu & \lambda_\nu & \lambda_{\nu+1} & \lambda_{\nu+2} & \lambda_{\nu+3} & \lambda_{\nu+4} & \lambda_{\nu+5} \\
1 & -0.0009 & 14869 & 48096 & 09608 & 28903 & 00000 \\
2 & -0.1225 & 51715 & 92326 & 00000 & 00000 & 00000 \\
3 & -0.3224 & 38869 & 40428 & 33333 & 00000 & 00000 \\
4 & -0.2320 & 47777 & 63333 & 00000 & 00000 & 00000 \\
5 & -0.1956 & 74316 & 61271 & 00000 & 00000 & 00000 \\
6 & -0.1393 & 30520 & 91922 & 18903 & 00000 & 00000 \\
\end{array}
\]
\[ m = 5; \quad \Pi_9(x) = \sum_{\nu=0}^{8} c_\nu x^\nu \quad \quad 1 = c_0 = c_8 \]
\[ 502 = c_1 = c_7 \]
\[ 14 \quad 608 = c_2 = c_6 \]
\[ 88 \quad 234 = c_3 = c_5 \]
\[ 156 \quad 190 = c_4 \]

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<tr>
<td>2</td>
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\[ m = 6; \quad \Pi_{11}(x) = \sum_{\nu=0}^{10} c_\nu x^\nu \quad \quad 1 = c_0 = c_{10} \]
\[ 2 \quad 036 = c_1 = c_9 \]
\[ 152 \quad 637 = c_2 = c_8 \]
\[ 2 \quad 203 \quad 488 = c_3 = c_7 \]
\[ 9 \quad 738 \quad 114 = c_4 = c_6 \]
\[ 15 \quad 724 \quad 248 = c_5 \]
ON SEMICARDINAL QUADRATURE FORMULAE

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\[
m = 7: \quad \prod_{k=0}^{12} c_k \cdot x^k = x^m + c_0 + c_1 x + \cdots + c_9 x^9 + c_{10} x^{10} + c_{11} x^{11}
\]

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8. The Numerical Values of $h_0^{(m)} = H_0^{(m)} - \frac{1}{3}$, $h_i^{(m)} = H_i^{(m)} - 1$ $(j \geq 1)$, for $m = 2, 3, \cdots, 7$. We have written the Q.F. (11) in the form (5.8), (5.9), where the coefficients $h_j^{(m)}$ are defined by (5.7). Below we give the values of the coefficients $C, C_1, \cdots, C_{m-1}$, appearing in the formulae (5.10), (5.11), which were used throughout the computation. The corresponding $\lambda_i$, for each $m$, are known from Section 7.

\[
\begin{array}{cccccccc}
\text{m = 2: } & C &=& -1.0566 & 24327 & 02594 \\
& C_1 &=& -1.50000 & 00000 & 00000 \\
\hline
j & 10^9 \cdot h_j^{(2)} & 10^9 \cdot h_j^{(2)} & 10^9 \cdot h_j^{(2)} & 10^9 \cdot h_j^{(2)} \\
0 & -105662433 & -2577388 & -13286 & -68 \\
1 & 133974596 & 690609 & 3560 & 13 \\
2 & -35898385 & -185048 & -954 & -5 \\
3 & 9618943 & 49583 & 256 & 15 \\
\end{array}
\]

\[
\begin{array}{cccccccc}
\text{m = 3: } & C &=& -1.55683 & 40723 & 44065 \\
& C_1 &=& 1.61253 & 86058 & 42966 \\
& C_2 &=& -0.69966 & 76766 & 67689 \\
\hline
j & 10^9 \cdot h_j^{(3)} & 10^9 \cdot h_j^{(3)} & 10^9 \cdot h_j^{(3)} & 10^9 \cdot h_j^{(3)} \\
0 & -14396343 & 919711 & -5267 & 21 \\
1 & 231765224 & -826580 & 2268 & 22 \\
2 & -126720028 & 355905 & -977 & 23 \\
3 & 55723001 & -153244 & 420 & 24 \\
4 & -24042963 & 65983 & -181 & \\
5 & 10354462 & -28411 & 78 & \\
6 & -4458469 & 12233 & -34 & \\
\end{array}
\]

\[
\begin{array}{cccccccc}
\text{m = 4: } & C &=& 29.79116 & 16580 & 89087 \\
& C_1 &=& -34.33080 & 08334 & 22275 \\
& C_2 &=& 5.03831 & 17952 & 59740 \\
& C_3 &=& -1.16658 & 41207 & 39341 \\
\end{array}
\]
### ON SEMICARDINAL QUADRATURE FORMULAE

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$m = 5$: \quad C = -1185.60066 \quad 69187 \quad 87416$

\[ C_1 = 1278.39413 \quad 47945 \quad 01574 \]
\[ C_2 = -104.82413 \quad 90602 \quad 21726 \]
\[ C_3 = 13.49945 \quad 93367 \quad 63573 \]
\[ C_4 = -2.15378 \quad 46634 \quad 43702 \]

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\[ C_3 = -291.24484 \quad 63712 \quad 04503 \]
\[ C_4 = 34.93429 \quad 00662 \quad 62594 \]
\[ C_5 = -4.25089 \quad 68338 \quad 22148 \]

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\[ C_2 = -1 \quad 81492.08505 \quad 99971 \quad 63019 \]
\[ C_3 = 9205.14045 \quad 15342 \quad 90528 \]
\[ C_4 = -806.75362 \quad 55949 \quad 48760 \]
\[ C_5 = 89.55157 \quad 21836 \quad 35045 \]
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## ON SEMICARDINAL QUADRATURE FORMULAE

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Mathematics Research Center  
University of Wisconsin  
Madison, Wisconsin 53706

Department of Mathematics  
Cleveland State University  
Cleveland, Ohio 44115


