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Reduction of the Pseudoinverse of a Hermitian Persymmetric Matrix

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Abstract. When the pseudoinverse of a Hermitian persymmetric matrix is computed, both computer time and storage can be reduced by taking advantage of the special structure of the matrix.

For any matrix M, let M' and M^* denote its transpose and conjugate transpose, respectively. Let J be a permutation matrix whose elements along the southwest-northeast diagonal are ones and whose remaining elements are zeros. Note that

$$J = J^* = J^{-1}.$$

Definition 1. M is *persymmetric* if $JM^*J = \overline{M}$, the complex conjugate of M. Note that all *Toeplitz* matrices ($t_{ij} = t_{i+1,j+1}$) are persymmetric.

Definition 2. M is *centrosymmetric* if $JMJ = M$; *skew-centrosymmetric* if $JMJ = -M$.

Note that if a persymmetric matrix is symmetric, it is centrosymmetric; if a persymmetric matrix is skew ($M' = -M$) it is skew-centrosymmetric. It is clear, therefore, that the real and imaginary parts of a *Hermitian persymmetric* matrix are centrosymmetric and skew-centrosymmetric, respectively.

In [2], matrix forms for the pseudoinverse of a centrosymmetric matrix are given in terms of the pseudoinverses of smaller matrices. Similar matrix forms for the pseudoinverse of a skew-centrosymmetric matrix are given in [3]. In this paper, we show that the pseudoinversion of a Hermitian persymmetric matrix reduces to the pseudoinversion of a real symmetric matrix of the same order.

Definition 3. The pseudoinverse A^+ of any matrix A is uniquely defined by the matrix equations:

$$AA^+A = A, \quad A^+AA^+ = A^+, \quad (A^+A)^* = A^+A, \quad (AA^+)^* = AA^+. \tag{1}$$

It is straightforward to verify that

$$A^+ = A^{-1} \qquad (A \text{ nonsingular}), \tag{2}$$

$$(UAV)^+ = V^*A^+U^* \qquad (U, V \text{ unitary}), \tag{3}$$

$$\overline{A}^+ = \overline{A^+}, \tag{4}$$

$$(A^*)^+ = (A^+)^*, \tag{5}$$

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and

$$D^+ = \operatorname{diag}(d_1, \ldots, d_n) \qquad (D \text{ diagonal}),$$

(6)
$$\begin{aligned} d_i &= 1/D_{ii} \qquad \text{if } D_{ii} \neq 0, \\ &= 0 \qquad \text{otherwise,} \end{aligned}$$

satisfy (1).

If P is an even order Hermitian persymmetric matrix that is split into real and imaginary parts, it may be partitioned as

(7)
$$P = \begin{pmatrix} K & HJ \\ JH & JKJ \end{pmatrix} + i \begin{pmatrix} S & NJ \\ -JN & -JSJ \end{pmatrix},$$

where K, H, N are real and symmetric, and S is real and skew. (Note that any complex centrosymmetric (skew-centrosymmetric) matrix of even order can be written in the partitioned form of the real (imaginary) part in (7) with K and H (S and N) complex.)

The pseudoinverse of P may be partitioned in the same form:

(8)
$$P^+ = \begin{pmatrix} B & CJ \\ JC & JBJ \end{pmatrix} + i \begin{pmatrix} F & GJ \\ -JG & -JFJ \end{pmatrix},$$

because it is also Hermitian persymmetric by (5), (3), (4) and Definition 1. The form of (7) suggests applying P to matrices of special form.

Let U, V be real matrices conformable with K such that

$$T = \begin{pmatrix} U \\ JU \end{pmatrix} + i \begin{pmatrix} V \\ -JV \end{pmatrix}$$

is nonsingular. Then, by [1],

$$PT = T\Lambda \qquad (\Lambda \text{ diagonal})$$

if and only if

(9)
$$Q\tilde{T} = \tilde{T}\Lambda,$$

where

(10)
$$Q = \begin{pmatrix} K + H & -(S - N) \\ S + N & K - H \end{pmatrix}, \qquad \tilde{T} = \begin{pmatrix} U \\ V \end{pmatrix}.$$

Note that Q is real and symmetric.

Now, suppose $\tilde{T}$ is orthogonal and satisfies (9). Then $T^*T = TT^* = 2I$. Thus, $P = 0.5T\Lambda T^*$ and, by direct substitution into (1),

(11)
$$P^+ = 0.5T\Lambda^+ T^*, \qquad Q^+ = \tilde{T}\Lambda^+ \tilde{T}^*.$$

Hence, with P^+ defined by (8), $P^+T = T\Lambda^+$ and, by [1],

(12)
$$Q_1 \tilde{T} = \tilde{T}\Lambda^+,$$

where

$$Q_1 = \begin{pmatrix} B + C & -(F - G) \\ F + G & B - C \end{pmatrix}. \tag{13}$$

But $Q_1 = Q^+$ by (11) and (12).

Thus, P^+ can be obtained by computing B, C, F, and G from Q^+ with a reduction in both storage and computer time.

If P is real and symmetric, then

$$N = S = 0$$

and

$$Q^+ = \operatorname{diag}((K + H)^+, (K - H)^+) = \operatorname{diag}(B + C, B - C).$$

In [2], the pseudoinversion of an arbitrary even order centrosymmetric matrix in the partitioned form of the real part of (7) is reduced also to the pseudoinversion of the matrices $K + H$ and $K - H$.

If P is pure imaginary, then

$$K = H = 0$$

and

$$Q^+ = \begin{pmatrix} 0 & (S + N)^+ \\ -(S - N)^+ & 0 \end{pmatrix} = \begin{pmatrix} 0 & -(F - G) \\ F + G & 0 \end{pmatrix}.$$

In [3], the pseudoinversion of an arbitrary even order skew-centrosymmetric matrix in the partitioned form of the imaginary part of (7) is reduced also to the pseudoinversion of the matrices $S + N$ and $S - N$.

When the order of P is odd, the analogous forms are

$$P = \begin{pmatrix} K & c & HJ \\ c' & \rho & c'J \\ JH & Jc & JKJ \end{pmatrix} + i \begin{pmatrix} S & d & NJ \\ -d' & 0 & d'J \\ -JN & -Jd & -JSJ \end{pmatrix},$$

$$Q = \begin{pmatrix} K + H & \sigma c & -S + N \\ \sigma c' & \rho & \sigma d' \\ S + N & \sigma d & K - H \end{pmatrix} \qquad (\sigma = \sqrt{2}),$$

where c, d are real column vectors conformable with J.

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