Negative Integral Powers of a Bidiagonal Matrix

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Abstract. The elements of the inverse of a bidiagonal matrix have been expressed in a convenient form. The higher negative integral powers of the bidiagonal matrix exhibit an interesting property: the \((ij)\)th element of the \((-m)\)th power is equal to the product of the corresponding element of the inverse by a Wronski polynomial, viz., the complete symmetric function of degree \((m - 1)\) of the diagonal elements, \(d_i, d_{i+1}, \ldots, d_n\), of the inverse matrix.

1. Introduction. Positive integral powers of a bidiagonal matrix with a fixed diagonal element \(b\) and superdiagonal element 1, have been reported by Varga [1]. In the present note, we shall find the negative integral powers of a general \(n \times n\) bidiagonal matrix \(B\), having diagonal elements \(b_i, i = 1, 2, \ldots, n\), and superdiagonal elements \(c_j, j = 1, 2, \ldots, n - 1\).

One may express \(B = (I - \Gamma)D^{-1}\), where \(I\) is the identity matrix and \(D^{-1}\) a diagonal matrix composed of the diagonal elements of \(B\). \(\Gamma\) is null, except for the elements \(\Gamma_{i,i+1} = -c_i/b_{i+1}\), for \(i = 1, 2, \ldots, n - 1\), on its first superdiagonal. The powers of \(\Gamma\) can easily be evaluated. In fact, the nonzero elements of \(\Gamma^m\) are given by

\[
(\Gamma^m)_{i,i+m} = \prod_{k=i}^{i+m-1} (-c_k/b_{k+1}), \quad \text{for} \ i = 1, 2, \ldots, n - m,
\]

occurring only on the \(m\)th superdiagonal.

The inverse \(E_1\) of \(B\) may be calculated either by the usual method of cofactors, or from the following expansions:

\[
E_1 = B^{-1} = D(I - \Gamma)^{-1} = D[I + \Gamma + \Gamma^2 + \Gamma^3 + \cdots + \Gamma^{n-1}].
\]

The elements of \(E_1\) may be written in a convenient form as follows:

\[
\begin{align*}
(1a) \quad e_1(i,j) &= 0 \quad \text{for} \ i > j, \\
(1b) \quad &= 1/b_j = d_j \quad \text{for} \ i = j, \\
(1c) \quad &= d_i \prod_{k=i}^{j-1} (-c_k/b_{k+1}) \quad \text{for} \ i < j.
\end{align*}
\]

The inverse is upper triangular but is not necessarily bidiagonal.

2. Powers of the Inverse. The product of \(E_1\) with itself is a matrix \(E_2\), which is also upper triangular. Elements of \(E_2\) are given by

\[
e_2(i,j) = e_1(i,j) \sum_{k=i}^{j} [d_k] \quad \text{for} \ i \leq j.
\]

\[
= 0 \quad \text{for} \ i > j.
\]
Result (2) may be generalized. In fact, the nth power of $E_1$ is an upper triangular matrix $E_n$ where the $(i,j)$th element, for $i \leq j$, is given by

$$e_n(i,j) = e_1(i,j) \sum_{k_1=i}^{j} \sum_{k_2=i}^{k_1} \sum_{k_3=i}^{k_2} \cdots \sum_{k_{n-2}=i}^{k_{n-3}} \sum_{k_{n-1}=i}^{k_{n-2}} [d_{k_1}d_{k_2}d_{k_3} \cdots d_{k_{n-2}}d_{k_{n-1}}].$$

**Proof.** Let us assume that result (3) is true for $n = m$.

$$e_{m+1}(i,j) = \sum_{k_0=i}^{j} [e_m(i,k_0)e_1(k_0,j)],$$

the other terms in the summation for $1 \leq k_0 \leq i - 1$ and $j + 1 \leq k_0 \leq n$, are zero, as both $e_m(p,q)$ and $e_1(p,q)$ are zero for $p > q$.

By writing the expression for $e_m(i,k_0)$ from result (3), which is assumed to be valid for $n = m$, we have

$$e_{m+1}(i,j) = \sum_{k_0=i}^{j} [e_1(i,k_0)e_1(k_0,j)]$$

$$\cdot \sum_{k_0=i}^{j} \sum_{k_2=i}^{k_1} \sum_{k_3=i}^{k_2} \cdots \sum_{k_{n-2}=i}^{k_{n-3}} \sum_{k_{n-1}=i}^{k_{n-2}} [d_{k_0}d_{k_1}d_{k_2}d_{k_3} \cdots d_{k_{n-2}}d_{k_{n-1}}].$$

The first summation is done by Eq. (2) and the expression reduces to

$$e_{m+1}(i,j) = e_1(i,j) \sum_{k_0=i}^{j} [d_{k_0}] \cdot \sum_{k_0=i}^{j} \sum_{k_2=i}^{k_1} \sum_{k_3=i}^{k_2} \cdots \sum_{k_{n-2}=i}^{k_{n-3}} \sum_{k_{n-1}=i}^{k_{n-2}} [d_{k_0}d_{k_1}d_{k_2}d_{k_3} \cdots d_{k_{n-2}}d_{k_{n-1}}].$$

After grouping the summations together, we find that the result is true for $n = m + 1$.

It has already been found true for $n = 2$ in Eq. (2), and therefore, by mathematical induction, we have the proof.

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