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A Generating Function for Triangular Partitions

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To D. H. Lehmer on his seventieth birthday

Abstract. Let $T_k(n)$ denote the number of solutions in nonnegative integers a_i, of the equation

$$n = \sum_{i=1}^{k} \sum_{j=1}^{k-i+1} a_{ij}$$

where the a_{ij} satisfy the inequalities $a_{ij} \geqslant a_{i+1,j}$, $a_{ij} \geqslant a_{i+1,j-1}$. We show that

$$\sum_{n=1}^{\infty} T_k(n)x^n = (1-x)^{-k}(1-x^3)^{-k+1}(1-x^5)^{-k+2} \cdots (1-x^{2k-1})^{-1}.$$

1. Introduction. We consider the triangular array of nonnegative integers (a_{ij})

$$T_k: \quad \begin{array}{ccccccc} a_{11} & & a_{12} & & a_{13} & \cdots & a_{1k} \\ & a_{21} & & a_{22} & \cdots & a_{2,k-1} & \\ & & a_{31} & & a_{32} & \cdots & \\ & & & \cdots & & & \\ & & & a_{k1} & & & \end{array} \tag{1.1}$$

satisfying the following system of inequalities:

$$a_{ij} \geqslant a_{i+1,j}, \qquad a_{ij} \geqslant a_{i+1,j-1}. \tag{1.2}$$

If in addition, the a_{ij} satisfy

$$\sum_{i+j \leqslant n+1} a_{ij} = n, \tag{1.3}$$

we call T_k a triangular partition of n of order k.

Let $T_k(n)$ denote the number of arrays T_k satisfying (1.2) and (1.3). Clearly

$$T_k(0) = 1 \qquad (k = 1, 2, 3, \cdots). \tag{1.4}$$

Since

$$T_1(n) = 1 \qquad (n = 0, 1, 2, \cdots), \tag{1.5}$$

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it follows at once that

$$\sum_{n=0}^{\infty} T_1(n)x^n = \frac{1}{1-x}. \tag{1.6}$$

For $k = 2$ we have

$$\sum_{n=0}^{\infty} T_2(n)x^n = \sum_{n=0}^{\infty} x^n \sum_{a+b+c=n;a\geqslant c,\, b\geqslant c} 1$$

$$= \sum_{a\geqslant b,\, c\geqslant b} x^{a+b+c} = \sum_{a,b,c=0}^{\infty} x^{a+b+3c},$$

so that

$$\sum_{n=0}^{\infty} T_2(n)x^n = \frac{1}{(1-x)^2(1-x^3)}. \tag{1.7}$$

Since $(1-x)^{-2}(1-x^3)^{-1} = \Sigma_{r=0}^{\infty}(r+1)x^r\,\Sigma_{s=0}^{\infty}x^{3s}$, it follows that $T_2(n) = \Sigma_{3s\leqslant n}(n-3s+1)$. Hence, if $m = [n/3]$, we get

$$T_2(n) = \tfrac{1}{2}(m+1)(2n-3m+2). \tag{1.8}$$

For $k = 3$ we find that

$$\sum_{n=0}^{\infty} T_3(n)x^n = (1-x)^{-3}(1-x^3)^{-2}(1-x^5)^{-1}. \tag{1.9}$$

For $k = 4$ we have

$$\sum_{n=0}^{\infty} T_4(n)x^n = (1-x)^{-4}(1-x^3)^{-3}(1-x^5)^{-2}(1-x^7)^{-1}. \tag{1.10}$$

The formulas (1.6), (1.7), (1.9), (1.10) suggest the general result

$$\sum_{n=0}^{\infty} T_k(n)x^n = (1-x)^{-k}(1-x^3)^{-k+1}(1-x^5)^{-k+2}\cdots(1-x^{2k-1})^{-1}. \tag{1.11}$$

The direct proof of (1.10) is rather tedious; the corresponding proof in the case $k = 5$ has not been completely carried out. We shall accordingly prove the general result (1.11) by an entirely different method which makes use of known results concerning MacMahon's theorem on k-line partitions [4, p. 243].

Put

$$\frac{1}{(1-x)(1-x^3)\cdots(1-x^{2k-1})} = \sum_{n=0}^{\infty} q_k(n)x^n,$$

so that $q_k(n)$ is the number of partitions of n into the parts $1, 3, 5, \cdots, 2k-1$, repetitions allowed. Then (1.11) yields the recurrence

$$T_k(n) = \sum_{j=0}^{n} q_k(j)T_{k-1}(n-j). \tag{1.12}$$

This evidently implies

$$T_k(n) = \sum q_k(j_1)q_{k-1}(j_2) \cdots q_2(j_{k-1}), \tag{1.13}$$

where the summation is over all nonnegative $j_1, j_2, \cdots, j_{k-1}$ satisfying $j_1 + j_2 + \cdots + j_{k-1} \leqslant n$.

Formulas (1.12) and (1.13) are indeed equivalent to (1.11). Thus a combinatorial proof of either (1.12) or (1.13) would yield a combinatorial proof of (1.11).

Another result equivalent to (1.11) is the following:

$$T_k(n) = \sum \prod_{j=1}^{k} \binom{k-j+n_j-1}{n_j}, \tag{1.14}$$

where the outer summation is over all nonnegative $n_1, n_2, \cdots, n_k$ satisfying $n_1 + 3n_2 + 5n_3 + \cdots + (2k-1)n_k = n$.

2. **Special Cases.** We shall now sketch the proof of (1.9). To begin with, it follows from the definition that

$$\sum_{n=0}^{\infty} T_3(n)x^n = (1-x^6)^{-1} \sum_{n=0}^{\infty} T_3'(n)x^n, \tag{2.1}$$

where $T_3'(n)$ denotes the number of arrays

$$\begin{array}{ccccc} a & & b & & c \\ & d & & e & \end{array}$$

satisfying $a \geqslant d,\ b \geqslant d,\ b \geqslant e,\ c \geqslant e$ and $a+b+c+d+e = n$. It follows that

$$\begin{aligned}
\sum_{n=0}^{\infty} T_3'(n)x^n &= \sum_{d,e=0}^{\infty} x^{d+e} \sum_{a,b,c;a\geqslant d,b\geqslant d;b\geqslant e,c\geqslant e} x^{a+b+c} \\
&= \sum_{d,e=0}^{\infty} x^{2d+2e} \sum_{a,c=0}^{\infty} x^{a+c} \sum_{b\geqslant d,b\geqslant e} x^b \\
&= (1-x)^{-2} \sum_{b=0}^{\infty} x^b \sum_{d=0}^{b} \sum_{e=0}^{b} x^{2d+2e} = (1-x)^{-2} \sum_{b=0}^{\infty} x^b \left(\frac{1-x^{2b+2}}{1-x^2}\right)^2 \\
&= (1-x)^{-2}(1-x^2)^{-2} \left\{ \frac{1}{1-x} - \frac{2x^2}{1-x^3} + \frac{x^4}{1-x^5} \right\} \\
&= \frac{1+x^3}{(1-x)^3(1-x^3)(1-x^5)}.
\end{aligned} \tag{2.2}$$

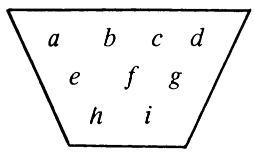
Substituting from (2.2) in (2.1), we get (1.9).

The proof of (1.10) is a good deal more involved and we give only a brief out-line. To begin with, we have

$$\sum_{n=0}^{\infty} T_4(n)x^n = (1 - x^{10})^{-1} \sum_{n=0}^{\infty} T_4'(n)x^n, \tag{2.3}$$

where $T_4'(n)$ denotes the number of arrays

(2.4)



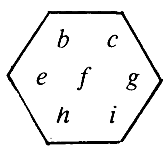
satisfying

$$a \geqslant e,\ b \geqslant e,\ b \geqslant f,\ c \geqslant f,\ c \geqslant g,\ d \geqslant g,\ e \geqslant h,\ \ f \geqslant h,\ f \geqslant i,\ g \geqslant i$$

and $a + b + \cdots + h + i = n$. In the next place we remove the corners on the top line of (2.4) to get

$$\sum_{n=0}^{\infty} T_4'(n)x^n = (1 - x)^{-2} \sum x^{b+c+2e+f+2g+h+i}, \tag{2.5}$$

where the summation on the right is over all arrays



satisfying

$$b \geqslant e,\quad b \geqslant f,\quad c \geqslant f,\quad c \geqslant g,\quad e \geqslant h,\quad f \geqslant h,\quad f \geqslant i,\quad g \geqslant i.$$

Thus we get for the sum on the right of (2.5)

$$(1 - x^2)^{-2} \sum_{f=0}^{\infty} x^f \left\{ \frac{x^{2f}}{(1 - x)^2} \left(\frac{1 - x^{3f+3}}{1 - x^3} \right)^2 - \frac{2x^{4f+2}}{(1 - x)(1 - x^3)} \, \frac{1 - x^{3f+3}}{1 - x^3} \, \frac{1 - x^{f+1}}{1 - x} \right.$$
$$\left. + \frac{x^{6f+4}}{(1 - x^3)^2} \left(\frac{1 - x^{f+1}}{1 - x} \right)^2 \right\}.$$

This reduces to

$$(1 + x^5)/(1 - x)^2 (1 - x^3)^3 (1 - x^5)(1 - x^7). \tag{2.6}$$

Hence, combining (2.3), (2.5) and (2.6), we get (1.10).

3. Restatement of Problem. It will be convenient to modify the original statement of the problem. Let A_n denote the set of lattice points in the first quadrant defined by

(3.1) $$A_n = \{(i,j) \mid i \geqslant 0,\ j \geqslant 0,\ i+j<n\}.$$

A_n is partially ordered if we put

(3.2) $$(i,j) \leqslant (i',j') \rightleftarrows i \leqslant i' \quad \text{and} \quad j \leqslant j'.$$

A nonnegative integer-valued function f defined on A_n will be called *increasing* if, for every $a, b \in A_n$, we have

(3.3) $$a \leqslant b \Rightarrow f(a) \leqslant f(b).$$

If f is increasing and takes on only the values 0 and 1, we may associate with f the subset A_f of A_n defined by

(3.4) $$a \in A_f \rightleftarrows f(a) = 1.$$

The collection of such subsets will be denoted by L_n. Note that L_n is a lattice with respect to union and intersection of sets. We show that L_n contains

(3.5) $$C_{n+2} = \frac{1}{n+2}\binom{2n+2}{n+1}$$

sets; C_n is a so-called Catalan number (for references see [1], [3]).

If f is increasing on A_n we put

(3.6) $$\sigma(f) = \sum_{a \in A_n} f(a)$$

and

(3.7) $$Q_n(x) = \sum x^{\sigma(f)},$$

where the summation is over all nonnegative integer-valued increasing functions on A_n. Clearly

(3.8) $$Q_n(x) = \sum_{N=0}^{\infty} T_n(N) x^N,$$

where $T_n(N)$ is the partition function defined in the introduction.

We remark, that if we define

$$\bar{Q}_n(x) = \sum x^{\sigma(f)} y^{\max f}$$

and replace A_n by

$$B_n = \{(i,j) \mid 0 \leqslant i < n,\ 0 \leqslant j < n\}.$$

then we are led to MacMahon's theorem for plane partitions.

4. The Lattice L_n. For every $A \in L_n$, let g_A denote the function defined by

(4.1) $$g_A(i) = \text{card}\,\{(n-i,j) \mid (n-i,j) \in A_n - A\} \quad (i = 0, 1, \cdots, n).$$

Note that

(i) g_A is increasing, and

(ii) $0 \leqslant g_A(i) \leqslant i \quad (i = 0, 1, \cdots, n)$.

Moreover, if A and B are in $\mathcal{L}_n$, then

$$g_{A\cup B} = \min(g_A, g_B), \qquad g_{A\cup B} = \max(g_A, g_B). \tag{4.2}$$

Let $\mathcal{F}_n$ consist of all integer-valued functions satisfying (i) and (ii). Then $\mathcal{F}_n$ is a lattice with respect to min and max. We summarize these observations in the following theorem.

THEOREM 1. *The lattices* $\mathcal{L}_n$ *and* $\mathcal{F}_n$ *are anti-isomorphic and contain*

$$C_{n+2} = \frac{1}{n+2}\binom{2n+2}{n+1} \tag{4.3}$$

elements.

PROOF. We show first that if $f \in \mathcal{F}_n$, then $f = g_A$ for some $A \in \mathcal{L}_n$. Let $f \in \mathcal{F}_n$ and put

$$A = \{(i,j) \mid f(n-i) \leqslant j\} \cap \mathcal{A}_n .$$

Now suppose $(i_0, j_0) \in A$ and both $(i_0 + 1, j)$ and $(i_0, j_0 + 1) \in \mathcal{A}_n$. Then

$$f(n - i_0 - 1) \leqslant f(n - i_0) \leqslant j_0, \qquad f(n - i_0) \leqslant j_0 < j_0 + 1,$$

so both $(i_0 + 1, j_0)$ and $(i_0, j_0 + 1) \in A$. Hence $A \in \mathcal{L}_n$ and

$$\begin{aligned} g_A(n - i_0) &= \operatorname{card}\{j \mid (i_0, j) \in \mathcal{A}_n - A\} \\ &= \operatorname{card}\{j \mid f(n - i_0) > j,\ j \geqslant 0\} = f(n - i_0). \end{aligned}$$

This, together with the previous remarks, shows $\mathcal{L}_n$ and $\mathcal{F}_n$ are indeed anti-isomorphic. It is well known (see for example [3]) that the number of elements in $\mathcal{F}_n$ is given by (4.3).

We note, for later use, that

$$|A| + \sum_{i=0}^{n} g_A(i) = |\mathcal{A}_n| = \tfrac{1}{2}n(n+1). \tag{4.4}$$

5. Chains in $\mathcal{L}_n$. By a *chain* in $\mathcal{L}_n$ we will mean any finite or infinite sequence of sets $A_i \in \mathcal{L}_n$ satisfying

$$A_i \subseteq A_{i+1} \qquad (i = 0, 1, 2, \cdots). \tag{5.1}$$

We will say that the chain $\{A_i\}_0^k$ *begins* at ϕ and *ends* at A_n if $A_0 = \phi$ and $A_k = A_n$.

There is a 1-1 correspondence between the set of increasing functions bounded by r on $\mathcal{A}_n$ and the chains $\{A_i\}_0^{r+1}$ in $\mathcal{L}_n$ which begin at ϕ and end at A_n. This correspondence is given by

$$A_i = \{a \mid f(a) \geqslant r - i + 1\} \qquad (i = 0, 1, \cdots, r+1). \tag{5.2}$$

It is clear that

$$\sigma(f) = |A_1| + |A_2| + \cdots + |A_r|, \tag{5.3}$$

where

$$\sigma(f) = \sum_{a \in \mathsf{A}_n} f(a). \tag{5.4}$$

Transferring the sets A_i to functions in F_n by the anti-automorphism of Theorem 1, we obtain

THEOREM 2. *There is a* 1-1 *correspondence between the set of increasing functions bounded by* r *on* A_n *and sequences of functions* $\{f_i\}_0^{r+1}$ *from* F_n *satisfying*

$$f_0 \geqslant f_1 \geqslant f_1 \geqslant \cdots \geqslant f_r \geqslant 0; \qquad f_0(x) = x. \tag{5.5}$$

Moreover

$$\sigma(f) = \tfrac{1}{2} m(n+1) - \sum_{i=1}^{r} \sum_{j=0}^{n} f_i(j). \tag{5.6}$$

PROOF. Follows from Theorem 1 and (4.4).

Another relation between increasing functions on A_n and chains in L_n is given as follows. Call a chain $\{A_i\}$ *proper* if $A_0 \neq \phi$ and $A_i \neq A_{i+1}$. Suppose f is an increasing function on A_n assuming the distinct nonzero values

$$t_1, t_1 + t_2, \cdots, t_1 + t_2 + \cdots + t_j; \qquad t_i > 0.$$

Let

$$B_i = \{a \mid f(a) \geqslant t_1 + \cdots + t_{j-i}\} \qquad (i = 0, 1, \cdots, j-1). \tag{5.7}$$

Then we have

$$\sigma(f) = t_1 |B_{j-1}| + t_2 |B_{j-2}| + \cdots + t_j |B_0|.$$

Hence the following theorem is immediate.

THEOREM 3. *The generating function*

$$Q_n(x) = \sum x^{\sigma(f)} \qquad (f \text{ increasing on } \mathsf{A}_n)$$

is given by

$$Q_n(x) = 1 + \sum \frac{x^{|B_0|}}{1 - x^{|B_0|}} \cdots \frac{x^{|B_j|}}{1 - x^{|B_j|}}, \tag{5.8}$$

where the summation is taken over all proper chains in L_n.

6. Computation of $Q_n(x)$. By Theorem 2 there is a 1-1 correspondence between increasing functions on A_n bounded by r and $n \times r$ arrays $\{f_j(i)\}$ satisfying

$$0 \leqslant f_j(1) \leqslant f_j(2) \leqslant \cdots \leqslant f_j(n) \qquad (j = 1, 2, \cdots, r) \tag{6.1}$$

and

$$i \geqslant f_1(i) \geqslant f_2(i) \geqslant \cdots \geqslant f_r(i) \geqslant 0 \qquad (i = 1, 2, \cdots, n). \tag{6.2}$$

Let $Q_n^{(r)}(x)$ denote the partition generating function for such arrays, that is,

$$Q_n^{(r)}(x) = \sum x^{\Sigma_{i,j} f_j(i)}, \tag{6.3}$$

where the outer sum is taken over all $\{f_j(i)\}$ satisfying (6.1) and (6.2). Specializing formula (6.12) of [2], we get

$$\begin{aligned} Q_n^{(r)}(x) &= x^{\frac{1}{2}n(n+1)} \left| x^{\frac{1}{2}(i-j)(i-j-1)} \begin{bmatrix} n-j+r \\ r-i+j-1 \end{bmatrix} \right| \\ &= x^{\frac{1}{2}n(n+1)} \left| x^{\frac{1}{2}(i-j)(i-j+1)} \begin{bmatrix} r+j-1 \\ 2j-i \end{bmatrix} \right| \qquad (i, j = 1, 2, \cdots, n), \end{aligned} \tag{6.4}$$

where

$$\begin{bmatrix} k \\ j \end{bmatrix} = \frac{(x)_k}{(x)_j (x)_{k-j}}, \qquad (x)_k = (1-x)(1-x^2)\cdots(1-x^k). \tag{6.5}$$

Replacing x by x^{-1}, it is easily verified that

$$\begin{bmatrix} k \\ j \end{bmatrix} \longrightarrow x^{j(j-k)} \begin{bmatrix} k \\ j \end{bmatrix}.$$

Thus we get

$$Q_n^{(r)}\left(\frac{1}{x}\right) = x^{-\frac{1}{2}rn(n+1)} \left| x^{\frac{1}{2}(i-j)^2} \begin{bmatrix} j+r-1 \\ 2j-i \end{bmatrix} \right| \qquad (i, j = 1, 2, \cdots, n).$$

By (5.6), we have

$$Q_n(x) = \lim_{r \to \infty} x^{\frac{1}{2}rn(n+1)} Q_n^{(r)}\left(\frac{1}{x}\right)$$

and therefore

$$Q_n(x) = \left| \frac{x^{(i-j)^2}}{(x)_{2j-i}} \right| = \left| \frac{x^{(i-j)^2}}{(x)_{2i-j}} \right| \qquad (i,\ j = 1, 2, \cdots, n). \tag{6.6}$$

It is convenient to put

$$D_k = \left| x^{(i-j)^2} \begin{bmatrix} 2i \\ j \end{bmatrix} \right| \qquad (i, j = 1, 2, \cdots, n), \tag{6.7}$$

so that (6.6) becomes

$$Q_n(x) = \frac{(x)_1 (x)_2 \cdots (x)_n}{(x)_2 (x)_4 \cdots (x)_{2n}} D_n. \tag{6.8}$$

We shall now evaluate D_k. Let R_i denote the ith row of D_k. We shall replace R_k by

$$\bar{R}_k = R_k - x\begin{bmatrix}k\\1\end{bmatrix}' R_{k-1} + x^2\begin{bmatrix}k\\2\end{bmatrix}' R_{k-2} - \cdots + (-1)^{k-1}a^{k-1}\begin{bmatrix}k\\k-1\end{bmatrix}' R_1,$$

where

$$\begin{bmatrix}k\\j\end{bmatrix}' = \frac{(x^2)'_k}{(x^2)'_j(x^2)'_{k-j}}, \qquad (a)'_k = (1-a)(1-x^2a)\cdots(1-x^{2k-2}a).$$

Then the jth element in $\bar{R}_k$ is equal to

$$\begin{aligned} r_j &= \sum_{s=0}^{k-1}(-1)^s x^s\begin{bmatrix}k\\s\end{bmatrix}' a^{(k-s-j)^2}\begin{bmatrix}2k-2s\\j\end{bmatrix} \\ &= \sum_{s=0}^{k}(-1)^{k-s}x^{k-s}\begin{bmatrix}k\\s\end{bmatrix}' x^{(s-j)^2}\begin{bmatrix}2s\\j\end{bmatrix}. \end{aligned}$$

Since

$$\begin{bmatrix}2s\\j\end{bmatrix} = \frac{1}{(x)_j}\sum_{t=0}^{j}(-1)^t x^{\frac{1}{2}t(t+1)+t(2s-j)}\begin{bmatrix}j\\t\end{bmatrix},$$

we get

$$\begin{aligned} r_j &= \frac{1}{(x)_j}\sum_{s=0}^{k}(-1)^{k-s}x^{k-s}\begin{bmatrix}k\\s\end{bmatrix}' x^{(s-j)^2}\sum_{t=0}^{j}(-1)^t x^{\frac{1}{2}t(t+1)+t(2s-j)}\begin{bmatrix}j\\t\end{bmatrix} \\ &= \frac{x^{j^2+k}}{(x)_j}\sum_{t=0}^{j}(-1)^t x^{\frac{1}{2}t(t+1)-tj}\begin{bmatrix}j\\t\end{bmatrix}\sum_{s=0}^{k}(-1)^{k-s}x^{s^2-s}\begin{bmatrix}k\\s\end{bmatrix}' x^{-2s(j-t)} \\ &= (-1)^k\frac{x^{j^2+k}}{(x)_j}\sum_{t=0}^{j}(-1)^t x^{\frac{1}{2}t(t+1)-tj}\begin{bmatrix}j\\t\end{bmatrix}(x^{-2(j-t)})'_k. \end{aligned}$$

Since

$$(x^{-2t})'_k = \begin{cases} 0 & (0 \leqslant t < k), \\ (-1)^k x^{-k(k+1)}(x^2)'_k & (t = k), \end{cases}$$

it follows that $r_j = 0$ for $0 \leqslant j < k$, while

$$r_k = (x^2)'_k/(x)_k = (1+x)(1+x^2)\cdots(1+x^k).$$

Hence

$$D_k = (1+x)(1+x^2)\cdots(1+x^k)D_{k-1}.$$

Since

$$D_1 = \left| \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right| = 1 + x,$$

we get

(6.9) $$D_k = (1 + x)^k (1 + x^2)^{k-1} \cdots (1 + x^k).$$

Substitution from (6.9) in (6.8) yields

THEOREM 4. *We have*

(6.10) $$Q_n(x) = \frac{1}{(1 - x)^n (1 - x^3)^{n-1} \cdots (1 - x^{2n-1})}.$$

This completes the proof of (1.11).

7. Number of Maximal Proper Chains in L_n. As an application of Theorem 4 we have the following.

THEOREM 5. *The number of maximal proper chains in* L_n *is given by*

(7.1) $$M_n = \frac{(\frac{1}{2}n(n + 1))!}{1^n\, 3^{n-1}\, 5^{n-2} \cdots (2n - 1)}.$$

PROOF. By Theorem 4 we see that

(7.2) $$\lim_{x \to 1} (1 - x)^{\frac{1}{2}n(n+1)} Q_n(x) = (1^n\, 3^{n-1}\, 5^{n-2} \cdots (2n - 1))^{-1}.$$

On the other hand, by (5.8),

(7.3) $$\lim_{x \to 1} (1 - x)^{\frac{1}{2}n(n+1)} Q_n(x) = \frac{M_n}{(\frac{1}{2}n(n + 1))!}.$$

Comparison of (7.2) and (7.3) yields (7.1).

8. A Related Partition Problem. Let $T'_k(n)$ denote the number of triangular arrays (a_{ij}) $(1 \leqslant j \leqslant i \leqslant k)$ satisfying the inequalities $a_{ij} \geqslant a_{i+1,j}$, $a_{ij} \geqslant a_{i+1,j+1}$ and also

$$\sum_{i=1}^{k} \sum_{j=1}^{i} a_{ij} = n.$$

It can be shown that

(8.1) $$\sum_{n=0}^{\infty} T'_k(n) x^n = \frac{(x)_1 (x)_2 \cdots (x)_k}{(x)_2 (x)_4 \cdots (x)_{2k}} D'_k,$$

where

$$D'_k = \left| x^{\frac{1}{2}(i-j)(i-j-1)} \begin{bmatrix} 2i \\ j \end{bmatrix} \right| \qquad (i, j = 1, 2, \cdots, k).$$

The first few values of D'_k follow:

$$D_1' = 1 + x, \quad D_2' = (1 + x)(1 + x^2)^2,$$

$$D_3' = (1 + x)(1 + x^2)(1 + x^3)(1 + x^2 + x^3 + 2x^4 + x^5 + x^6 + x^8).$$

We remark that, when $k \longrightarrow \infty$, the generating function (8.1) reduces to the generating function for plane partitions.

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