Not Every Number is the Sum or Difference of Two Prime Powers

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Abstract. Every odd number less than 262144 is the sum or difference of a power of two and a prime. An interesting example is 113921 = \( p - 2^{141} \). Using covering congruences, we exhibit a 26-digit odd number which is neither the sum nor difference of a power of two and a prime. The method is then modified to exhibit an arithmetic progression of numbers which are not the sum or difference of two prime powers.

In 1950, P. Erdös [1] used covering congruences to exhibit numbers not of the form \( 2^n + p \). Using similar methods, we study the sequences \( 2^n + M \) and \( \pm(2^n - M) \) for a fixed integer \( M \). In particular, we prove

**Theorem 1.** There exists an arithmetic progression of odd numbers which are neither the sum nor difference of a power of two and a prime.

Via the proof of Theorem 1 and primality testing programs of M. Wunderlich, we show

**Corollary.** 47867742232066880047611079 is prime and neither the sum nor difference of a power of two and a prime.

A modification of the proof of Theorem 1 yields

**Theorem 2.** There exist odd numbers which are neither the sum nor difference of a power of two and a prime power.

Curiously, numbers satisfying Theorem 1 appear relatively difficult to find. We have calculated (on an IBM 360) that if \( M < 2^{18} \) and \( M \) is odd, then \( M \) is the sum or difference of a power of two and a prime. The \( M \) for which it is most difficult to verify this fact is 113921. Here 113921 + 2^{141} is prime (proved using Lucas sequences) and 141 is the smallest \( n \) such that \( |113921 \pm 2^n| \) is prime.

**Proof of Theorem 1 and its Corollary.** Recall that a collection of congruences \( n \equiv b_i(h_i) \) is called a covering if each integer satisfies at least one of the congruences. We exhibit two coverings of the integers. For each congruence in either cover, we require \( M \) to satisfy a related congruence. Any \( M \) which satisfies these related congruences enjoys the following property: There is a list of 18 primes such that for any nonnegative integer \( n, M + 2^n \equiv 0(p_i) \) and \( M - 2^n \equiv 0(p_i) \) for some \( p_i \) and some \( p_i \) in our list. Theorem 1 follows directly; the number in the corollary satisfies the requisite congruences on \( M \).
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Congruences on $M$

$M + 2^n \equiv 0(p_i) \iff n \equiv b_i(h_i)$

$M + 2^1 \equiv 0(3)$

$M + 2^0 \equiv 0(5)$

$M + 2^6 \equiv 0(17)$

$M + 2^{10} \equiv 0(13)$

$M + 2^2 \equiv 0(97)$

$M + 2^{10} \equiv 0(257)$

$M + 2^{18} \equiv 0(241)$

$M \equiv 2^n(p_i) \iff n \equiv b_i(h_i)$

$M \equiv 2^0(3)$

$M \equiv 2^0(7)$

$M \equiv 2^{17}(109)$

$M \equiv 2^{35}(37)$

$M \equiv 2^{11}(19)$

$M \equiv 2^5(73)$

$M \equiv 2^5(331)$

$M \equiv 2^1(41)$

$M \equiv 2^3(61)$

$M \equiv 2^2(31)$

$M \equiv 2^3(11)$

$M \equiv 2^4(151)$

End Theorem 1 proof.

We remark that it is impossible to cover both $M + 2^n$ and $M - 2^n$ with primes less than 331. More precisely, for any $M$ there is an $n$ such that either $M + 2^n$ or $M - 2^n$ has all its prime factors greater than 330. On the other hand, by using more primes we can dispense with 3; that is, we can find an $M$ and a set of 42 primes so that $3M + 2^n$ and $3M - 2^n$ are simultaneously covered.

Proof of Theorem 2. We find additional conditions on $M$ and additional primes (16 in all).

First we will insure that all terms, $M + 2^n$, divisible by $3^3$ are divisible by 37 or 109. Put $M + 2^{17} \equiv 0(3^3)$ and thus $3^3$ divides $M + 2^n$ when $n \equiv 17(18)$. To cover $n \equiv 17(36)$ we put $M + 2^{17} \equiv 0(37)$ and to cover $n \equiv 35(36)$ we put $M + 2^{35} \equiv 0(109)$.

Next we add conditions on the remaining primes covering $M + 2^n$.

$M + 2^8 \equiv 0(5^2 \cdot 11)$

$M + 2^6 \equiv 0(17^2 \cdot 137)$

$M + 2^2 \equiv 0(13^2 \cdot 53)$

$M + 2^{10} \equiv 0(97^2 \cdot 389)$

$M + 2^{18} \equiv 0(241^2 \cdot 1447)$

$M + 2^{10} \equiv 0(257 \cdot 673)$

Note that we have used our cover from Theorem 1 with the positions of 13 and 97 interchanged.
Now notice that \( M + 2^n \) can never be a prime power.

Consider \( |M - 2^n| \). First we have \( M \equiv 2^8(3^2) \) from above, and each term with \( n = 8(6) \) is either \( n = 8(12) \) or \( n = 2(4) \). But since \( M \equiv 2^8(13) \) and \( M \equiv 2^2(5) \) from above, we have a second prime factor whenever \( 3^2 \) divides \( |M - 2^n| \). Finally, we add conditions for the remaining primes covering \( |M - 2^n| \), noticing that the conditions for 11, 37, and 109 are consistent with those already described.

\[
\begin{align*}
M &= 2^3(7^2 \cdot 43) \quad \text{(note } M \equiv 2^9(3 \cdot 7)) \\
M &= 2^{17}(109^2 \cdot 2617) \\
M &= 2^{35}(37^2 \cdot 149) \\
M &= 2^{11}(19^2 \cdot 571) \\
M &= 2^5(73^2 \cdot 439) \\
M &= 2^{25}(331^2 \cdot 1987) \\
M &= 2^1(41^2 \cdot 83) \\
M &= 2^{31}(61 \cdot 1321) \\
M &= 2^2(31^3 \cdot 311) \\
M &= 2^3(11^3 \cdot 23) \\
M &= 2^{19}(151^2 \cdot 907) \quad \text{(note } M \equiv 2^4(3 \cdot 151))
\end{align*}
\]

Hence \( |M - 2^n| \) has at least two distinct prime factors provided that it is greater than 331. End Theorem 2 proof.

A routine use of Lehmer’s linear equation solver shows that 6120 6699060672 7677809211 5601756625 4819576161 6319229817 3436854933 4512406741 7420946855 8999326569 satisfies these congruences.

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