REVIEWS AND DESCRIPTIONS OF TABLES AND BOOKS

The numbers in brackets are assigned according to the revised indexing system printed in Volume 28, Number 128, October 1974, pages 1191–1194.

1 [7].—Daniel Shanks & John W. Wrench, Jr., Sums of Reciprocals to 1,000,000, 1961, ms. of 20 computer sheets deposited in the UMT file.

Herein are tabulated values of the partial sums $\sum_{1}^{N} n^{-1}$ of the harmonic series for $N = 10^{4}(10^{4})10^{6}$, truncated to 1060D. These were computed on an IBM 7090 system at the same time that we evaluated $\pi$ [1] and $e$ [2], and they were intended to be used by the second author in computing Euler’s constant, $\gamma$, by means of the Euler-Maclaurin formula. However, Knuth [3] computed $\gamma$ to higher precision before this was completed.

For the sake of comparison we list these sums truncated to 1000D for $N = 10^{4}$, $10^{5}$, and $10^{6}$, respectively, with *(Af)* denoting the omission of $M$ digits:

$$
9.78760603604438226417 *(960)* 92164466197618373424,
12.09014612986342794736 *(960)* 76024520048801442625,
14.39272672286572363138 *(960)* 34360832668760078693.
$$

Our value corresponding to $N = 10^{4}$ agrees in its entirety with the value found to 1275D by Knuth, which has been deposited in the UMT file along with his unpublished table of Bernoulli numbers mentioned on p. 277 of [3].

D. S., J. W. W.


2 [9].—David Ballew, Janell Case & Robert N. Higgins, Table of $\phi(n) = \phi(n + 1)$, South Dakota School of Mines and Technology, 1974, ii + 3 pages, deposited in the UMT file.

There are listed here the 88 solutions of $\phi(n) = \phi(n + 1)$ from $n = 3$ to $n = 2792144$. (Previous tables have listed $n = 1$ also; counting this, there are 89 solutions for $n < 2.8 \cdot 10^{6}$.) This extends the tables of the 36 solutions to $n = 10^{5}$ by Lal and Gillard [1] and the 56 solutions to $n = 5 \cdot 10^{5}$ by Miller [2]. Note that Miller is wrong in stating that the next solution is $n = 525986$. She has omitted $n = 524432$. 329
A propos my editorial note to [2], there is only one further case in this extension (if I did it correctly). For \( n = 2539004 \), multiplication \((\text{mod } n)\) is isomorphic to multiplication \((\text{mod } n + 1)\). That is a much more stringent requirement; I do not know if anyone has made a heuristic estimate of whether there are infinitely many such \( n \).

D. S.


The table in the appendix lists the class number \( H \) and fundamental unit \( \epsilon_0 \) \((0 < \epsilon_0 < 1)\) of the pure cubic fields \( Q(\rho) \) where \( \rho = D^{1/3} \). For each cube-free \( D \) between 2 and 998 there is listed \( H, U, V, W, T, \) and \( J \) where

\[
\epsilon_0 = \frac{(U + V\rho + W\rho^2)}{T}
\]

and \( J \) is the length of the period of Voronoi's algorithm. The largest \( U \) here is a 330-decimal number for \( D = 951 \) where \( H = 1 \). Here, \( J = 1352 \), and for large \( U \) one finds that \( J / \log_{10} U \approx 4.1 \). Presumably, the mean value of this ratio is analogous to Lévy's constant but its identity is not known to me. The largest \( H \) equals 162 here for \( D = 813 \). Some fields are given twice: e.g., \( Q((12)^{1/3}) = Q((18)^{1/3}) \) and so its \( \epsilon_0 \) is given in two forms. Happily, the \( H \) then agree—in all cases that I checked.

A direct comparison with Wada's units to \( D = 249 \), see [1], is not possible since Wada gives the reciprocal \( \epsilon = 1/\epsilon_0 = (A + B\rho + C\rho^2)/E \) instead. It is of some interest to argue which unit is preferable. Usually, \( U, V, W \) have only one-half the decimals of \( A, B, C \); for example, for \( D = 239 \), \( U \) has 94 decimals while \( A \) has 188. But for applications, \( \epsilon \) is usually preferable. Thus, in evaluating the regulator

\[
R = |\log \epsilon_0|,
\]

the formula (1) can suffer catastrophic loss of significance since \( \epsilon_0 \) may be exceedingly small. Of course, one can obtain \( \epsilon \) from \( \epsilon_0 \) by

\[
\epsilon = \frac{(U^2 - DVW) + (W^2D - UV)\rho + (V^2 - UW)\rho^2}{T}
\]

if \( T = 1 \). So, for such large \( U, V, W, R = \log (3U^2 - 3DVW) \) will be very accurate.

The text describes Voronoi's algorithm and refers to earlier, less extensive tables by Markov, Cassels, Selmer, etc.

D. S.


These are the tables referred to repeatedly in Brent’s paper [1]. The numbers \(\pi(n), \pi_2(n)\) and \(B^*(n)\) and

\[ r_i(n), s_i(n), R_i(n, n'), \rho_i(n, n') \]

for \(i = 1, 2, 3\) are defined in [1]. They are listed in Table 1 for 533 values of \(n\):

\[10^4 (10^4) 10^5 (10^5) 10^7 (10^6) 10^8 (10^7) 10^9 (10^8) 10^{10} (10^9) 83 \cdot 10^9.\]

Table 2 (1 page long) lists \(n, \pi_2(n), B(n), \) and \(B^*(n)\) with some auxiliary functions for

\[10^5 (10^5) 10^6 (10^6) 10^7 (10^7) 10^8 (10^8) 10^9 (10^9) 10^{10} (10^{10}) 8 \cdot 10^{10}.\]

The author indicates that he has much more detailed tables and is continuing to \(10^{11}\).

Section 3 of [1] ends with the same conclusion given earlier in our [2]: that the unpredictable fluctuations of \(\pi_2(n)\) around the Hardy-Littlewood approximation makes it difficult to compute Brun’s constant accurately. But his Fig. 3 allows for a posteriori judgment; although we do not know where \(s_3(n)\) is going, we know where it’s been!

We see that Fröberg’s low value at \(\log_{10} n = 6.02\), our high value at \(\log_{10} n = 7.51\) and Bohman’s low value at \(\log_{10} n = 9.30\) all correlate (inversely) with the peaks and valleys of Fig. 3. In fact, Fig. 3 between \(\log_{10} n = 6.63\) and 7.19 gives a crude, distorted, upside-down version of our Fig. 1 [2] and \(\log_{10} n\) between 7.19 and 7.51 continues with our Fig. 2. Thus, for Brun’s constant, it does appear that \(n = 8 \cdot 10^{10}\) is a good time to quit since \(s_3(n)\) is then very small.

Concerning the negative peaks in Brent’s Fig. 1 at \(\log_{10} n = 8.04\) and 8.25, it would be nice to know when they are exceeded. As Brent is aware, if a likely \(n\) were known that is not too large, one could restart his tables of \(r_i(n)\) and \(s_i(n)\) for \(i = 1, 2\) by computing a fiducial mark \(\pi(n)\) by Lehmer’s method.

D. S.


The Kummer Sum

\[
S_p = \sum_{n=0}^{p-1} \exp\left(\frac{2\pi in^3}{p}\right) = 1 + 2 \sum_{n=1}^{(p-1)/2} \cos\left(\frac{2\pi n^3}{p}\right)
\]
for a prime \( p \equiv 1 \pmod{3} \) equals one of the three real roots of

\[
x^3 = 3px + pA
\]

where \( 4p = A^2 + 27B^2, A \equiv 1 \pmod{3} \). On the basis of only the 45 primes \( p < 500 \), Kummer conjectured that \( S_p \) occurs as the minimum, median, or maximum root of (2) in the proportions: 1, 2, 3. Subsequent work of von Neumann [1] and Emma Lehmer [2] suggested that as \( p \to \infty \) there may be equidistribution instead, and Vinogradov once thought [3] that he had proven this.

Fröberg [4] computed \( S_p \) for the 8988 \( p < 2 \cdot 10^5 \) and found 2370, 2990, and 3628 solutions, respectively, with the maximal roots now down to 40.4%, the minimal roots up to 26.4% and the median roots remaining very close to 33%. There is deposited here a listing of these 8988 primes: \( p, A, B, S \) (to 6D), and an asterisk in the appropriate column labelled MIN, MED, MAX. \( S \) has rounding errors (example below) but this accuracy is not needed here since it suffices to know where \( S_p \) lies in the three intervals: \( I_1 < -\sqrt{p} < I_2 < +\sqrt{p} < I_3 \). Note also that it is unnecessary to compute \( A \) and \( B \) separately, since \( A = \frac{1}{p} (3pS - 3p) \).

After extrapolating the three empirical percentage functions \( \%(P) \), for \( p < P \), according to the proposed formulas

\[
\%(P) = a + b \exp(-cP),
\]

Fröberg conjectures that the asymptotic proportions are 4, 5, 6—that is, that the limiting percentages are \( 26\frac{2}{3}, 33\frac{1}{3}, \) and \( 40\% \), respectively. This reviewer is skeptical for two reasons: (A) No rationale, even heuristic, is given to support (3) and the exponential there tends to leave the purported asymptotic values \( a \) near his final empirical values at \( P = 2 \cdot 10^5 \). Whereas, any logarithmic function in place of (3) would make equidistribution more plausible. (B) If 4, 5, 6 are the true asymptotic proportions, it should be possible to find some reasonably simple heuristic argument that suggests these proportions. I know of none.

There are 51 cases here with \( A > 0, B = 1 \). Here the two smaller roots are nearly equal, being approximately \( -\sqrt{p} + 1\frac{1}{2} \), while the largest root is nearly \( +\sqrt{p} \). If there is a difference in the ultimate proportion of MIN and MAX one might expect to see it here since the dissymmetry is maximized. One does not; there are 16, 18, and 17 cases, respectively. In the 53 cases with \( A < 0, B = 1 \), there is the opposite dissymmetry with the two larger roots close together near \( \sqrt{p} + 1\frac{1}{2} \). One now finds 16, 19, 18 cases. (For more on the cyclic cubic fields with \( B = 1 \), see [5].) In the 74 cases here with \( A = +1, -2, +4 \) or \( -5 \), where the median root is \( \approx -A/3 \) while the extreme roots are \( \approx \pm \sqrt{3p} \), one has the greatest symmetry. Here one finds 24, 21, and 29 cases. These are all small numbers but they seem to suggest equidistribution; certainly nothing here suggests that the MAX are 50\% more numerous than the MIN. But if there is equidistribution, why are the MAX more common when \( p \) is small? A good, quantitative explanation is wanted.
|S_p| is bounded below by 1/3. The smallest S_p here is one of the aforementioned A = 1, namely, p = 170647, A = 1, B = 159, S_p = -0.3333334056. (The table lists S_p = -0.335414 for this p, showing that four decimals are corrupted in adding up the 85 thousand cosines.) The existence of such small S_p illustrates the marked distinction between these cubic sums and the quadratic Gauss Sums with n^2 instead of n^3 in (1). Then, |S_p| = √p, as is well known. For other recent work, see Cassels [6] and the references cited there.

D. S.


For any product m = p_1 ⋅ p_2 ⋅ ⋯ ⋅ p_n of distinct primes p ≡ 1 (mod 3) there are 2^{n-1} distinct cyclic cubic fields of discriminant m^2 and for m = 9 ⋅ p_1 ⋅ ⋯ ⋅ p_n there are 2^n such fields. Altogether there are 630 fields with m < 4000. Table 1 lists each such m with (A) its prime decomposition; (B) its appropriate representation 4m = a^2 + 27b^2; (C) its class number h; and, in most cases, (D) tr(e) and tr(e^{-1}). These latter integers give the equation

x^3 = tr(e)x^2 − tr(e^{-1})x + 1

satisfied by the fundamental units and having a discriminant m^2k^2 for some index k ≥ 1. When tr(e) and tr(e^{-1}) are too large, they are omitted here since they were not obtained with the precision used. (These large units are only missing from Table 1 for some cases of h = 1 or 3 when ζ_k/ζ(1) is relatively large because one or more small primes split in the field. The first units missing are those for m = 919 which has h = 1 and both 2 and 3 as splitting primes.)

This table, and those that follow, were computed by a new, interesting method described in Marie Gras’s paper [1]. The tables are more easily extended to larger m by this method if h is large. There are known criteria for 9|h and 4|h, [2], [3]. Table 2 continues with 154 more m < 10^4 having 9|h while Table 3 contains 119 m < 10^4 having 4|h. These two tables overlap some. Sometimes, units are missing, as before.
Table 4 contains all \( m \) between \( 4 \cdot 10^3 \) and \( 2 \cdot 10^4 \) having a representation
\[ 4m = a^2 + 27 \text{ or } 1 + 27b^2 \text{ or } 9 + 27b^2. \]
In these 89 fields, \( \text{tr}(e) \) and \( \text{tr}(e^{-1}) \) are never missing since they are known a priori. They equal \( \pm 1/2(a \mp 3) \),
\( \pm 3/2(9b \mp 1) \) and \( \pm 3/2(3b \mp 1) \), respectively. These units are relatively small and
the class numbers, correspondingly, are relatively large. The largest is \( h = 129 \) for
\( m = 97 \cdot 181 = (1 + 27 \cdot 51^2)/4. \)
These tables of cyclic cubic fields go far beyond earlier tables of Hasse, Cohn
and Gorn, and Godwin. For the “simplest cubics”, having \( 4m = a^2 + 27 \), the re-
viewer has gone further [4] using an entirely different method.
D. S.

1. MARIE NICOLE GRAS, “Méthodes et algorithmes pour le calcul numérique du nombre
de classes et des unités des extensions cubiques cycliques de \( Q \),” Crelle’s J. (To appear.)
2. G. GRAS, “Sur les \( l \)-classes d’idéaux dans les extensions cycliques relatives de degré
premier \( l \),” Thèse, Grenoble, 1972.
3. MARIE-NICOLE MONTOUCHET, “Sur le nombre de classes du sous-corps cubique de
\( Q(p) \) \( (p \equiv 1(3)) \),” Thèse, Grenoble, 1971.

7 [9].—WELLS JOHNSON, The Irregular Primes to 30000 and Related Tables, ms. of
28 computer pages (+ 1 introductory page), deposited in the UMT file, June 1974.
This unpublished table constitutes an appendix to a paper published elsewhere in
this issue. The 13-column table presents the complete list of 1619 irregular pairs
\((p, 2k)\) with \( p < 30000 \) together with some computations which depend upon this
list. The table shows that Fermat’s Last Theorem is true for all prime exponents \( p < 30000 \). In addition, the tables of [1], [2], [3] are completed to 30000, so that the
cyclotomic invariants of Iwasawa are completely determined for primes within this
range. The computations were performed on the PDP-10 computer at Bowdoin Col-
lege.

AUTHOR’S SUMMARY

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3. W. JOHNSON, “Irregular prime divisors of the Bernoulli numbers,” Math. Comp., v. 28,
8 [9].—J. P. KULIK, *Magnus Canon Divisorum ⋯*, 8 ms. volumes (v. 2 now missing), deposited in the Library of the Academy of Sciences, Vienna in 1867.

A photostatic copy of that portion of v. 1 consisting of pages 260 through 416 has been deposited by D. H. Lehmer in the UMT file. This portion of Kulik's monumental table gives the least prime factor for all integers not divisible by 2, 3, or 5 between 9,000,000 and 12,642,600. The deposited copy includes handwritten corrections by Professor Lehmer inserted in the margins.

A detailed description of the complete table has been published by Joffe [1], superseding that of D. N. Lehmer [2].

In 1948 an announcement [3] was made that the Carnegie Institution of Washington had made in the preceding year a negative microfilm of this same portion of v. 1 and that it was prepared to supply positive microfilm copies at a nominal charge ($1.00 per copy at that time).

J. W. W.


9 [9].—SIGEKATU KURODA, *Table of Class Numbers, h(p) Greater than 1, for Fields Q(√p), p ≡ 1 (mod 4) ≤ 2776817*, University of Maryland, 1965, copy deposited in the UMT file.

The table consists of 88 Xeroxed computer sheets containing class numbers $h(p)$ for primes of the form $p = 4n + 1$. The purpose of the computation was not simply to calculate $h(p)$, but to test a conjectured method of doing so. It is well known that every ideal class of $Q(√p)$ contains an integral ideal with norm $< B = \frac{1}{2}√p$. Also the class number $h$ is odd, the nonprincipal classes (if $h > 1$) occurring in conjugate pairs $C_i, C_i' = C_i^{-1}$, $1 ≤ i ≤ h' = \frac{1}{2}(h - 1)$. The conjecture states that the $h'$ classes $C_i$ can be represented by integral ideals all having distinct norms less than $B$.

Thus, only those $p$ with $h(p) > 1$ were printed. There are $22 \cdot 2^{10} = 22528$ such primes up to $p = 2776817$. A printed count shows that altogether 100811 cases were computed; thus 78283 cases with $h = 1$ are omitted. The table lists the primes $p$ and the class numbers $h(p)$ on alternate pages. Every other page contains 512 primes in 16 columns and 32 rows, each position identified by a number in base 32. The class number $h(p)$ is found on the next sheet and in the same position as $p$, cf. [1]. The class numbers are written in base 32 and followed by a symbol ($P, Q, C, or D$) indicating that the conjecture was verified for that case. The primes $p$ were printed both in decimal and in base 32. The copy deposited contains the decimal version except for the first two pages. These were missing from the reviewer’s copy of the table.
and so are given in base 32. For statistics about the class number distribution see the reviewer’s paper in this issue [2].

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10 [9].—Richard B. Lakein, Class Numbers of 5000 Quartic Fields \( \mathbb{Q}(\sqrt{\pi}) \), SUNY at Buffalo, 1973, ms. of 21 computer sheets deposited in the UMT file.

Let \( P \) be a rational prime \( \equiv 1 \mod 8 \), and \( \pi = a + bi \) a Gaussian prime with norm \( a^2 + b^2 = P \), normalized so that \( a, b > 0, b \equiv 0 \mod 4 \). Then \( K = \mathbb{Q}(\sqrt{\pi}) \) is a totally complex quartic field with no quadratic subfield other than \( \mathbb{Q}(i) \). The arithmetic of \( K \) has many strong analogies to that of a real quadratic field with prime discriminant. In particular, the class number \( h(\pi) \) of \( K \) is odd.

This table lists the first 5000 primes \( P \equiv 1 \mod 8 \) (from \( P = 17 \) through \( P = 226241 \)), the (normalized) Gaussian prime factor \( \pi \) of \( P \), and the class number \( h(\pi) \) of the quartic field \( K = \mathbb{Q}(\sqrt{\pi}) \). The final page of the table lists the cumulative distribution of class numbers for each successive 1000 fields. The distribution of class numbers is very close to that for the first 5000 real quadratic prime discriminants [2]. Details of the method of calculation, as well as the class number distribution, are contained in [1].

Author’s summary


The absolute modular invariant \( j(\tau) \), defined by

\[
j(\tau) = x^{-1} \prod_{n=1}^{\infty} (1 - x^n)^{-24} \left\{ 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)x^n \right\}^3 = \sum_{n=-1}^{\infty} c(n)x^n = x^{-1} + 744 + 196884x + \cdots,
\]

where \( x = \exp 2\pi ir \) and \( \sigma_r(n) = \sum_{d|n} d^r \), is the Hauptmodul of the classical modular
group $\Gamma$. Its coefficients possess many remarkable arithmetic properties, which are set forth in the appended references. For example, the congruence

$$(n + 1)c(n) \equiv 0 \pmod{24},$$

due to D. H. Lehmer [2], implies that $c(n)$ is even except possibly when $n \equiv 7 \pmod{8}$. In this case it may be shown that $c(n)$ assumes both even and odd values infinitely often, although necessary and sufficient conditions for $c(n)$ to be odd are still unknown.

The coefficients were first computed for $-1 < n < 24$ by H. S. Zuckerman [7] and then for $-1 < n < 100$ by A. van Wijngaarden [6]. Here we tabulate the coefficients for $-1 < n < 500$. There would seem to be little point in extending the table further, since $c(500)$ is already a number of 120 digits.

The coefficients were computed, using residue arithmetic, by means of the following formula [5]:

$$c(n) = p_{-24}(n + 1) + \frac{65520}{691} \sum_{k=0}^{n} \sigma_{11}(k + 1)p_{-24}(n - k), \quad n \geq 1,$$

where $\sum_{n=0}^{\infty} p_{-24}(n)x^n = \prod_{n=1}^{\infty} (1 - x^n)^{-24}$.

The total computation time on a UNIVAC 1108 system was approximately four minutes.

**Author's Summary**


12 [9]—Daniel Shanks. *Table of the Greatest Prime Factor of $N^2 + 1$ for $N = 1(1)185000$, 1959*, two ms. volumes, each of 185 computer sheets, bound in cardboard covers and deposited in the UMT file.

This table was calculated in 1959 on an IBM 704 system by the $p$-adic sieve method described completely in [1]. The method is extraordinarily efficient: each division performed is known a priori to have a zero remainder. From the complete factorization of $n^2 + 1$ for $n = 1(1)185000$ I then tabulated only the greatest
prime factors, 500 per page, arranged in an obvious format. (One can see at once which

\[ n^2 + 1 \]

are prime by the relative size of the corresponding listed factors.)

These factorizations relate to questions concerning reducible numbers, primes of
the form \( n^2 + 1 \), formulas for \( \pi \), and other questions surveyed in [1].

In [2] and [3] similar \( p \)-adic sieves were run for \( n^2 \pm 2, n^2 \pm 3, n^2 + 4, n^2 \pm 5, \)
\( n^2 \pm 6, \) and \( n^2 \pm 7 \) for \( n = 1(1)180000 \), but only statistical information was pre-
served, not the complete table of greatest prime factors.

D. S.

1. DANIEL SHANKS, “A sieve method for factoring numbers of the form \( n^2 + 1 \),” MTAC,
v. 13, 1959, pp. 78–86.

2. DANIEL SHANKS, “On the conjecture of Hardy & Littlewood concerning the number of

3. DANIEL SHANKS, “Supplementary data and remarks concerning a Hardy-Littlewood con-

13 [9].—J. D. SWIFT, Table of Carmichael Numbers to \( 10^9 \), University of California
at Los Angeles, ms. of 20 pages, \( 8_{\frac{1}{2}}'' \times 11'' \), deposited in the UMT file.

A Carmichael number, CN, is a composite number \( n \) such that \( a^{n-1} \equiv 1 \)
(mod \( n \)) whenever \( (a, n) = 1 \). Carmichael numbers are starred in Poulet’s table [1]
of pseudoprimes less than \( 10^8 \). The present table corrects that table and extends the
range to \( 10^9 \). The CN’s are given with their prime factors.

Calculations were performed on a CDC 1604 made available by IDA, in Princeton.
The computer programs used depended explicitly on congruence properties of CN’s
with respect to their component primes rather than on the pseudoprimality with respect
to any particular base. A different routine was run for each possible number of primes
occurring in the factorization, from 3 (the absolute minimum) to 6 (the effective maxi-
mum defined by the upper limit of the table).

For example, consider \( n = p_1p_2p_3 = (r_1 + 1)(r_2 + 1)(r_3 + 1) \). The basic
criteria are that \( r_i | p_k - 1 \) where \( i, j, k \) is a permutation of 1, 2, 3. For a fixed
choice of \( p_1 \) (assuming \( p_1 < p_2 < p_3 \)), bounds on the limits of the calculation are
obtained. In this simplest case an explicit bound is available:

\[
p_1p_2p_3 \leq (p_1^6 + 2p_1^5 - p_1^4 - p_1^3 + 2p_1^2 - p_1)/2;
\]

and this is actually a CN for \( p_1 = 3, 5, 31, \cdots \) (?).

The total number of CN’s less than or equal to each power of 10 is as follows:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( \text{CN}(x) )</th>
<th>ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 10^4 )</td>
<td>7</td>
<td>2.3</td>
</tr>
<tr>
<td>( 10^5 )</td>
<td>16</td>
<td>2.7</td>
</tr>
<tr>
<td>( 10^6 )</td>
<td>43</td>
<td>2.4</td>
</tr>
<tr>
<td>( 10^7 )</td>
<td>105</td>
<td>2.4</td>
</tr>
<tr>
<td>( 10^8 )</td>
<td>255</td>
<td>2.4</td>
</tr>
<tr>
<td>( 10^9 )</td>
<td>646</td>
<td>2.5</td>
</tr>
</tbody>
</table>
The known results thus appear to suggest an asymptotic relation for $CN(x)$ as of the order of $Cx^{0.4}$, which is much smaller than has been conjectured by Erdös [2]. In this connection, the change from 2.43 to 2.53 in the ratios of the last orders of magnitude computed may be significant.

Author's summary


Table 1 gives the fundamental unit $e_0 = (U + Vp + Wp^2)/T$ for all irreducible cubics $p^3 = Qp + N$ having $|Q|, N \leq 50$ and a discriminant $D < 0$. Table 3 gives $e_0$ for $p^3 = Ap^2 + Bp + C$ with $A, |B|, |C| \leq 10$ and $D < 0$. For $D > 0$ there are two fundamental units and Tables 2 and 4 give both of them for the same range of $Q, N$ and $A, B, C$, respectively.

These are the most extensive tables of cubic units known to me although for special types, such as cyclic or pure cubic fields, units have been computed that are not included here.

No attempt is made here to identify different $Q, N$ or $A, B, C$ that give the same field. That would be a valuable addition, especially if it gave the transformation taking one $p$ into another.

D. S.

15 [9] — Kenneth S. Williams & Barry Lowe, Table of Solutions $(x, u, v, w)$ of the Diophantine System $16p = x^2 + 50u^2 + 50v^2 + 125w^2, xw = v^2 - 4uw - u^2, x \equiv 1 \pmod{5}$ for Primes $p < 10000, p \equiv 1 \pmod{5}$, Carleton University, Ottawa, 1974, manuscript of 13 pages deposited in the UMT file.

The authors tabulate the values $(x, u, v, w)$ of one of the four solutions of the Diophantine system in the title for all primes $p \equiv 1 \pmod{10}$ less than 10000, the remaining three solutions being $(x, -u, -v, w), (x, v, -u, -w)$, and $(x, -v, u, -w)$. These solutions are obtained from the coefficients of the Jacobi function of order five which have been tabulated by Tanner [1] for $p < 10000$. Two errors in Tanner's tables are noted and one in earlier tables.

A derivation of the well-known linear relationship between these coefficients (which are in fact Jacobsthal sums) and the solutions $x, u, v, w$ is also given.

It should be pointed out that Joseph Muskat has obtained the solutions $(x, u, v, w)$ a number of years ago for all $p \equiv 1 \pmod{10}$ for $p < 50000$ from
the corresponding values of the cyclotomic numbers of order five, which he also tabulated together with the ratios of $x^2$, $50u^2$, $50v^2$, $125w^2$ to $16p$. This comprises 26 computer sheets and could probably still be obtained from the author.

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