On Laplace Transforms Near the Origin

By R. Wong

Abstract. Let \( f(t) \) be locally integrable on \([0, \infty)\) and let \( \mathcal{L}\{f\}(s) \) denote the Laplace transform of \( f(t) \). In this note, we prove that if \( f(t) \sim t^{-\beta} \sum_{n=0}^{\infty} a_n (\log t)^{-n} \) as \( t \to \infty \), where \( 0 < \Re \beta < 1 \), then \( \mathcal{L}\{f\}(s) \sim s^{1-\beta} \sum_{n=0}^{\infty} c_n (\log 1/s)^{-n} \) as \( s \to 0 \) in \( |\arg s| < \pi/2 - \Delta \), the \( c_n \) being constants.

1. Introduction. Let \( f(t) \) be locally integrable on \([0, \infty)\) and let \( \mathcal{L}\{f\} \) denote the Laplace transform of \( f(t) \). That is,

\[
\mathcal{L}\{f\} = \int_{0}^{\infty} f(t)e^{-st}dt,
\]

whenever the integral on the right converges. In a recent paper, Handelsman and Lew [2] have studied the asymptotic behavior of \( \mathcal{L}\{f\} \) as \( s \to 0 \), when \( f(t) \) satisfies

\[
f(t) \sim \exp(-ct^p) \sum_{m,n=0}^{\infty} c_{mn} t^m (\log t)^n \quad \text{as} \quad t \to \infty,
\]

where \( p > 0 \), \( \Re c > 0 \), \( \Re r_m \downarrow -\infty \) as \( m \to \infty \), and the set \( \{n: c_{mn} \neq 0\} \) is finite for each \( m \). In this note, we consider the case

\[
f(t) \sim t^{-\beta} \sum_{n=0}^{\infty} a_n (\log t)^{-n} \quad \text{as} \quad t \to \infty,
\]

where \( 0 \leq \Re \beta < 1 \). Our result will complement that of Handelsman and Lew.

2. Main Theorem. For convenience, we introduce the notation

\[
\mathcal{L}_c\{f\} = \int_{c}^{\infty} f(t)e^{-st}dt
\]

where \( c \) is a fixed real number > 1. In [3], it was shown that for any complex number \( \beta \) with \( \Re \beta < 1 \),

\[
\mathcal{L}_c\{t^{-\beta}(\log t)^{-n}\} \sim s^{1-\beta} \sum_{r=0}^{\infty} \binom{-n}{r} \Gamma(r)(1-\beta) \left(\log \frac{1}{s}\right)^{-r}
\]

as \( s \to 0 \) in \( S(\Delta) \), where

Received July 23, 1973.

AMS (MOS) subject classifications (1970). Primary 41A60.

Key words and phrases. Laplace transform, asymptotic expansion, Ramanujan function.

* Research partially supported by the National Research Council of Canada under Contract No. A7359.

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Let
\[ S(\Delta) = \{ s : |\arg s| \leq \pi/2 - \Delta \}. \]

\[ c_n = \sum_{r=0}^{n} a_{n-r} \binom{r-n}{r} \Gamma(r)(1 - \beta). \]

We are now ready to state and prove our main result.

**Theorem.** If \( f(t) \) is locally integrable on \([0, \infty)\) and satisfies (1.3), then as \( s \to 0 \) in \( S(\Delta) \)

\[ L(f) \sim s^{\beta-1} \sum_{n=0}^{\infty} c_n \left( \log \frac{1}{s} \right)^{-n}, \]

where the coefficients \( c_n \) are given in (2.4).

**Proof.** For any \( c > 1 \),

\[ L(f) = L_c(f) + \int_0^c f(t)e^{-st} dt = L_c(f) + O(1) \]

as \( s \to 0 \) in \( S(\Delta) \).

Writing
\[ f(t) = \sum_{n=0}^{N} a_n t^{-\beta} (\log t)^{-n} + R_N(t) \]
gives
\[ L_c(f) = \sum_{n=0}^{N} a_n L_c(t^{-\beta} (\log t)^{-n}) + r_N, \]
where
\[ r_N = \int_c^\infty R_N(t)e^{-st} dt. \]

From (1.3), it follows that there are constants \( K > 0 \) and \( c > 1 \) such that

\[ |R_N(t)| \leq K |t^{-\beta} (\log t)^{-N-1}| \quad \text{for } t \geq c. \]

Hence, by (2.2),
\[ |r_N| \leq K \int_c^\infty t^{-\gamma} (\log t)^{-N-1}e^{-\sigma t} dt \]
\[ = O(\sigma^{-1}(\log \sigma)^{-N-1}) \quad \text{as } \sigma \to 0, \]

where \( \gamma = \Re \beta \) and \( \sigma = \Re s \). Since \( |s| \sin \Delta \leq \sigma \leq |s| \) for any \( s \in S(\Delta) \), (2.11) is equivalent to

\[ r_N = O(|s|^{-1}(\log |s|)^{-N-1}) = O(s^{-1}(\log s)^{-N-1}) \]

as \( s \to 0 \) in \( S(\Delta) \). Coupling the results (2.8) and (2.12), we obtain

\[ L_c(f) = \sum_{n=0}^{N} a_n L_c(t^{-\beta} (\log t)^{-n}) + O(s^{-1}(\log s)^{-N-1}) \]
as $s \to 0$ in $S(\Delta)$. Since the $O$-term in (2.6) may be included in that of (2.12), (2.13) implies

$$L(f) = \sum_{n=0}^{N} a_n C_n \{t^{-\beta} (\log t)^{-n} + O(s^{\beta-1} (\log s)^{-N-1})\}$$

as $s \to 0$ in $S(\Delta)$. In view of (2.2), each term in (2.14) can be expanded in powers of $(\log 1/s)^{-1}$. Hence, by regrouping the terms, we have for any $N > 0$

$$L(f) = s^{\beta-1} \left[ \sum_{n=0}^{N} A_n \left( \log \frac{1}{s} \right)^{-n} + O((\log s)^{-N-1}) \right]$$

as $s \to 0$ in $S(\Delta)$. This completes the proof of our theorem.

3. An Example. The Ramanujan function is defined by

$$N(s) = \int_0^\infty \frac{e^{-st}}{t^2 + (\log t)^2} \, dt.$$  

Recently, Bouwkamp [1] obtained the asymptotic expansion

$$N(s) \sim \sum_{n=0}^{\infty} c_n (\log s)^{-n-1} \quad \text{as } s \to \infty,$$

where the coefficients were determined by the generating function

$$\frac{1}{\Gamma(1-x)} = \sum_{n=0}^{\infty} c_n \frac{x^n}{n!}.$$  

Our aim is to find the asymptotic behavior of $N(s)$ as $s \to 0$.

Integrating by parts, we obtain from (3.1)

$$N(s) = \frac{1}{2} + \frac{s}{\pi} \int_0^\infty \tan^{-1} \left( \frac{1}{\pi} \log t \right) e^{-st} \, dt.$$  

The function $\tan^{-1} (\log t/\pi)$ has the convergent expansion

$$\tan^{-1} \left( \frac{1}{\pi} \log t \right) = \frac{\pi}{2} - \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \left( \frac{\pi}{\log t} \right)^{2n+1},$$

for $t > e^n$. Hence, the conditions of the theorem are trivially satisfied and we have

$$N(s) \sim 1 - \sum_{\nu=0}^{\infty} a_{\nu} \left( \log \frac{1}{s} \right)^{-\nu-1}$$

as $s \to 0$ in $S(\Delta)$, where

$$a_{\nu} = \sum_{2n+r=\nu} \frac{(-1)^n \pi^{2n}}{2n+1} \left( \frac{2n-1}{r} \right) \Gamma(r+1).$$

It is interesting to note that these coefficients are precisely the ones given by Bouwkamp for the asymptotic expansion of $N(s)$ as $s \to \infty$. To see this, we recall the identity
\[ \Gamma(x)\Gamma(1 - x) = \pi/\sin \pi x. \] Hence, from (3.3),

\begin{equation}
(3.8) \quad c_\nu = \nu! \sum_{2n + r = \nu} \frac{(-1)^n \pi^{2n}}{(2n + 1)!} \frac{\Gamma^{(r)}(1)}{r!}.
\end{equation}

Comparing (3.7) and (3.8), we have

\begin{equation}
(3.9) \quad a_\nu = (-1)^\nu c_\nu, \quad \nu = 0, 1, 2, \ldots.
\end{equation}

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