On Lower and Upper Bounds of the Difference Between the Arithmetic and the Geometric Mean

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Abstract. Lower and upper bounds of the difference between the arithmetic and the geometric mean of \( n \) quantities are given here in terms of \( n \), the smallest value \( a \) and the largest value \( A \) of given \( n \) quantities. Also, an upper bound for the difference, independent of \( n \), is given in terms of \( a \) and \( A \). All the bounds obtained are sharp.

1. Introduction. Let \( a_1, \ldots, a_n \) be \( n \) quantities such that \( 0 < a = a_1 < a_2 < \cdots < a_n = A \). Let \( A_n \) be their arithmetic mean, \( G_n \) their geometric mean. Trivial lower and upper bounds of the difference \( A_n - G_n \) are 0 and \( A - a \) respectively. A nice upper bound has been obtained in [2]. Here we shall prove the following inequalities:

\[
\frac{n-1}{n} (\sqrt[n]{A} - \sqrt[n]{a})^2 < A_n - G_n \leq ca + (1 - c)A - a^cA^{1-c}
\]

where

\[
c = \frac{\log[(A/(A-a))\log A/a]}{\log A/a}.
\]

The inequalities give lower and upper bounds of the difference \( A_n - G_n \) in terms of the smallest value \( a \) and the largest value \( A \) of the given \( n \) quantities. Instead of a discrete method, a continuous and analytic approach is used to obtain the inequalities.

2. Lower Bounds. We consider the lower bound of \( n \) quantities \( 0 < a = a_1 < \cdots < a_{k-1} < a_k+1 < \cdots < a_n = A \) with \( a_k = x, \;
1 < k < n, \) to be a variable in the interval \([a, A]\). Let the arithmetic and the geometric means of the fixed \( n - 1 \) quantities be \( A_{n-1} \) and \( G_{n-1} \), respectively. Then

\[
A_n - G_n = n^{-1} \{(n-1)A_{n-1} + x\} - \{G_{n-1}^n x\}^{1/n} = D_n(x).
\]

Since \( D'_n(x) = 0 \) at \( x = G_{n-1} \), the lower bound of \( D_n(x) \) for \( x \) in the interval \([a, A]\) is \( D_n(G_{n-1}) = ((n-1)/n)(A_{n-1} - G_{n-1}) \).

This result can also be found in [1, p. 12], but the method used here seems to be simpler and more straightforward. By repeating this process, we have

\[
A_n - G_n \geq \frac{n-1}{n} (A_{n-1} - G_{n-1}) \geq \frac{n-2}{n-1} (A_{n-2} - G_{n-2}) \geq \cdots \geq \frac{2}{n} (A_2 - G_2) = \frac{2}{n} \left( \frac{a + A}{2} - \sqrt{aA} \right) = \frac{1}{n} (\sqrt{A} - \sqrt{a})^2.
\]
Equality holds only if \( a_2 = \cdots = a_{n-1} = \sqrt{a_1 a_n} = \sqrt{a A} \).

3. Upper Bounds. Now we investigate the upper bound of \( A_n - G_n \). The maximum of \( D_n(x) \) on \([a, A]\) is attained at the endpoint \( a \) or \( A \). Thus, the maximum of \( A_n - G_n \) of \( n \) quantities is attained when \( a = a_1 = \cdots = a_k \leq a_{k+1} = \cdots = a_n = A \) for some \( k, \ 1 < k < n \), with the form

\[
\frac{k c + (n - k) A}{n} \left[ a^k A^{n-k} \right]^{1/n} = a \left[ k + (n-k) A \right]^{(n-k)/n}.
\]

For the sake of simplicity, let \( a = 1 \) and consider the function

\[
D(x) = \frac{x + (n-x) A}{n} - A^{(n-x)/n}.
\]

Through straight calculation, we have \( D'(x) = 0 \) for \( x = cn \), where

\[
c = \frac{\log[(A/(A - 1))\log A]}{\log A}.
\]

Thus, the upper bound for \( A_n - G_n \) is \( D(cn) = c + (1-c)A - A^{1-c} = U \) which is independent of \( n \). By repeated application of L'Hospital's rule, we have

\[
\lim_{A \to 1} c = \frac{1}{2} \quad \text{and} \quad \lim_{A \to \infty} c = 0.
\]

The upper bound is attained only at its limiting case \( A \to 1 \) or \( n \to \infty \). But this is the best possible bound independent of the positive integer \( n \geq 2 \). For a fixed \( n \), the sharp upper bound of \( A_n - G_n \) is attained by \( D(k_n) \) with \( k_n = [cn] \) or \([cn] + 1 \), where \([cn]\) denotes the largest integer not greater than \( cn \). Therefore, we have lower and upper bounds of \( A_n - G_n \), both dependent and independent of \( n \), in terms of \( a = 1 \) and \( A \) as follows:

\[
0 \leq n^{-1} (\sqrt{A} - 1)^2 \leq A_n - G_n \leq D(k_n) \leq c + (1-c)A - A^{1-c}.
\]

Some numerical data are shown below.

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