On Lower and Upper Bounds of the Difference Between the Arithmetic and the Geometric Mean

By S. H. Tung

Abstract. Lower and upper bounds of the difference between the arithmetic and the geometric mean of \( n \) quantities are given here in terms of \( n \), the smallest value \( a \) and the largest value \( A \) of given \( n \) quantities. Also, an upper bound for the difference, independent of \( n \), is given in terms of \( a \) and \( A \). All the bounds obtained are sharp.

1. Introduction. Let \( a_1, \ldots, a_n \) be \( n \) quantities such that \( 0 < a = a_1 \leq a_2 \leq \ldots \leq a_n = A \). Let \( A_n \) be their arithmetic mean, \( G_n \) their geometric mean. Trivial lower and upper bounds of the difference \( A_n - G_n \) are 0 and \( A - a \) respectively. A nice upper bound has been obtained in [2]. Here we shall prove the following inequalities:

\[
n^{-1}(\sqrt[n]{A} - \sqrt[n]{a})^2 < A_n - G_n < ca + (1 - c)A - a^cA^{1-c}
\]

where

\[
c = \frac{\log[(A/(A-a)]}{\log A/a}.
\]

The inequalities give lower and upper bounds of the difference \( A_n - G_n \) in terms of the smallest value \( a \) and the largest value \( A \) of the given \( n \) quantities. Instead of a discrete method, a continuous and analytic approach is used to obtain the inequalities.

2. Lower Bounds. We consider the lower bound of \( n \) quantities \( 0 < a = a_1 \leq \ldots \leq a_{k-1} \leq a_{k+1} \leq \ldots \leq a_n = A \) with \( a_k = x \), \( 1 < k < n \), to be a variable in the interval \([a, A]\). Let the arithmetic and the geometric means of the fixed \( n-1 \) quantities be \( A_{n-1} \) and \( G_{n-1} \), respectively. Then

\[
A_n - G_n = n^{-1} \left\{ (n-1)A_{n-1} + x \right\} - (G_{n-1}^{n-1}x)^{1/n} = D_n(x).
\]

Since \( D_n(x) = 0 \) at \( x = G_{n-1} \), the lower bound of \( D_n(x) \) for \( x \) in the interval \([a, A]\) is

\[
D_n(G_{n-1}) = ((n-1)/n)(A_{n-1} - G_{n-1}).
\]

This result can also be found in [1, p. 12], but the method used here seems to be simpler and more straightforward. By repeating this process, we have

\[
A_n - G_n \geq \frac{n-1}{n}(A_{n-1} - G_{n-1}) \geq \frac{n-1}{n-2}(A_{n-2} - G_{n-2}) \geq \ldots \geq \frac{2}{n}(A_2 - G_2) = \frac{2}{n} \left( \frac{a + A}{2} - \sqrt{aA} \right) = \frac{1}{n}(\sqrt{A} - \sqrt{a})^2.
\]
Equality holds only if \( a_2 = \cdots = a_{n-1} = \sqrt[n-1]{a_1 a_n} = \sqrt[n]{a A} \).

3. Upper Bounds. Now we investigate the upper bound of \( A_n - G_n \). The maximum of \( D_n(x) \) on \([a, A]\) is attained at the endpoint \( a \) or \( A \). Thus, the maximum of \( A_n - G_n \) of \( n \) quantities is attained when \( a \equiv a_1 = \cdots = a_k \leq a_{k+1} = \cdots = a_n \equiv A \) for some \( k, \ 1 < k < n \), with the form

\[
\frac{kc + (n-k)A}{n} = \frac{(n-k)}{n} \left( \frac{(n-k)}{kc} \right) = \frac{n-k}{k} \left( \frac{n-k}{c} \right) \frac{A}{n}
\]

For the sake of simplicity, let \( a = 1 \) and consider the function

\[
D(x) = \frac{x + (n-x)A}{n} - A^{n-x}/n.
\]

Through straight calculation, we have \( D'(x) = 0 \) for \( x = cn \), where

\[
c = \frac{\log[(A/(A-1))\log A]}{\log A}.
\]

Thus, the upper bound for \( A_n - G_n \) is \( D(cn) = c + (1-c)A - A^{1-c} = U \) which is independent of \( n \). By repeated application of L'Hospital's rule, we have

\[
\lim_{A \to 1} c = \frac{1}{2} \quad \text{and} \quad \lim_{A \to \infty} c = 0.
\]

The upper bound is attained only at its limiting case \( A \to 1 \) or \( n \to \infty \). But this is the best possible bound independent of the positive integer \( n \geq 2 \). For a fixed \( n \), the sharp upper bound of \( A_n - G_n \) is attained by \( D(k_n) \) with \( k_n = [cn] \) or \([cn] + 1 \), where \([cn]\) denotes the largest integer not greater than \( cn \). Therefore, we have lower and upper bounds of \( A_n - G_n \), both dependent and independent of \( n \), in terms of \( a = 1 \) and \( A \) as follows:

\[
0 \leq n^{-1}(\sqrt[n]{A} - 1)^2 \leq A_n - G_n \leq D(k_n) \leq c + (1-c)A - A^{1-c}.
\]

Some numerical data are shown below.

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