Properties of the Taylor Series Expansion Coefficients of the Jacobian Elliptic Functions

By Alois Schett

Abstract. Properties of the Taylor series expansion coefficients of the Jacobian elliptic functions and tables for the first fifteen leading terms are given. Relations of these coefficients with the randomization distributions are shown.

Little is known about the Taylor series expansion coefficients of the Jacobian elliptic functions $\text{sn}(u, k)$, $\text{cn}(u, k)$ and $\text{dn}(u, k)$. No recurrence formula exists for these coefficients. Only four to five leading terms of the series are given in literature ([1], [2]).

We present in this paper properties of these coefficients, show relations between them and randomization distributions [3, p. 51], and give tables for the first fifteen leading terms.

We consider the differential equations

$$
\begin{align*}
\frac{d}{du} y_1(u) - C_1 y_2(u) y_3(u) &= 0, \\
\frac{d}{du} y_2(u) - C_2 y_1(u) y_3(u) &= 0, \\
\frac{d}{du} y_3(u) - C_3 y_1(u) y_2(u) &= 0.
\end{align*}
$$

Solution functions of (1) for $C_1 = 1$, $C_2 = -1$, $C_3 = -k^2$ are the Jacobian elliptic functions $y_1 = \text{sn}(u, k)$, $y_2 = \text{cn}(u, k)$, $y_3 = \text{dn}(u, k)$ ([1], [2]).

The formal Taylor series of the functions $y_1, y_2, y_3$ read

$$
\begin{align*}
y_1(u) &= \sum_{n=0}^{\infty} \frac{(u-u_0)^n}{n!} \left[ \sum a_{j_1 j_2 j_3}^1 C_1^{j_1} C_2^{j_2} C_3^{j_3} y_1^{i_1} y_2^{i_2} y_3^{i_3} \right], \\
y_2(u) &= \sum_{n=0}^{\infty} \frac{(u-u_0)^n}{n!} \left[ \sum b_{h_1 h_2 h_3}^{1} C_1^{h_1} C_2^{h_2} C_3^{h_3} y_1^{s_1} y_2^{s_2} y_3^{s_3} \right], \\
y_3(u) &= \sum_{n=0}^{\infty} \frac{(u-u_0)^n}{n!} \left[ \sum c_{r_1 r_2 r_3}^{1} C_1^{r_1} C_2^{r_2} C_3^{r_3} y_1^{q_1} y_2^{q_2} y_3^{q_3} \right].
\end{align*}
$$

The summation over the indices $j_1, j_2, j_3; h_1, h_2, h_3; r_1, r_2, r_3$ and their relation to the exponents of $y_1, y_2, y_3$ are specified in Theorem 1.

$$
y_m = y_m(u_0) \quad (m = 1, 2, 3).
$$

For $u_0 = 0$, $y_1 = 0$, $y_2 = y_3 = 1$, this series is convergent in the region $|u| < K'$. [2], where

Received October 14, 1974; revised March 3, 1975 and May 27, 1975.


Copyright © 1976, American Mathematical Society
\[ K' = K(k') = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k'^2 \sin^2 \theta}}, \quad k' = \sqrt{1 - k^2}. \]

For the explicit series of (2), the elements \( a_{i_1i_2i_3}, b_{h_1h_2h_3}, c_{r_1r_2r_3} \) have to be determined and summation over the indices has to be specified.

**Theorem 1.** \( a_{i_1i_2i_3} \neq 0 \) only for
\[
\begin{align*}
J_1 &= \begin{cases} n/2 & \text{for } n \text{ even}, \\ (n+1)/2 & \text{for } n \text{ odd}, \end{cases} \\
J_2 &= \begin{cases} n/2 & \text{for } n \text{ even}, \\ (n+1)/n/2 & \text{for } n \text{ odd}, \end{cases} \\
J_3 &= \begin{cases} 0, 1, \ldots, J_3; & \text{for } n \text{ odd}, \\
& \text{for } n \text{ even}, \end{cases}
\end{align*}
\]

and \( i_1, i_2, i_3 \) satisfying the relation \( i_1 + i_2 + i_3 = n \). \( i_1, i_2, i_3 \) is obtained from the relations \( i_1 = n + 1 - 2j_1, \quad i_2 = n - 2j_2, \quad i_3 = n - 2j_3 \). \( b_{h_1h_2h_3} \neq 0 \) only for \( h_1 = i_3, h_2 = i_1, h_3 = i_2 \).

The exponents \( s_1, s_2, s_3 \) are related to \( h_1, h_2, h_3 \) as follows: \( s_1 = n - 2h_1, \quad s_2 = n + 1 - 2h_2, \quad s_3 = n - 2h_3 \). \( c_{r_1r_2r_3} \neq 0 \) only for \( r_1 = j_2, r_2 = j_3, r_3 = j_1 \). The exponents \( q_1, q_2, q_3 \) are defined by the relations \( q_1 = n - 2r_1, \quad q_2 = n - 2r_2, \quad q_3 = n + 1 - 2r_3 \).

For \( u_0 = 0 \), i.e.,
\[
y_1(u_0 = 0) = sn(0, k) = 0, \quad y_2(u_0 = 0) = cn(0, k) = 1, \quad y_3(u_0 = 0) = dn(0, k) = 1,
\]
we obtain
\[
a_{i_1i_2i_3} \neq 0 \quad \text{only for } j_1 = (n + 1)/2; j_2 = 0, 1, \ldots, (n-1)/2; n \text{ odd,} \\
b_{h_1h_2h_3} \neq 0 \quad \text{only for } h_3 = n/2; h_1 = 0, 1, \ldots, n/2 - 1; n \text{ even,} \\
c_{r_1r_2r_3} \neq 0 \quad \text{only for } r_2 = n/2; r_3 = 1, 2, \ldots, n/2; n \text{ even.}
\]

In the following table the elements \( a_{i_1i_2i_3}, b_{h_1h_2h_3}, c_{r_1r_2r_3} \) and the exponents \( i_1, i_2, i_3; s_1, s_2, s_3; q_1, q_2, q_3 \) are given as an illustration for \( n = 3 \) and \( n = 4 \).

**Table 1**

<table>
<thead>
<tr>
<th>( n = 3 )</th>
<th>( a_{i_1i_2i_3} )</th>
<th>( i_1 )</th>
<th>( i_2 )</th>
<th>( i_3 )</th>
<th>( b_{h_1h_2h_3} )</th>
<th>( h_1 )</th>
<th>( h_2 )</th>
<th>( h_3 )</th>
<th>( s_1 )</th>
<th>( s_2 )</th>
<th>( s_3 )</th>
<th>( c_{r_1r_2r_3} )</th>
<th>( r_1 )</th>
<th>( r_2 )</th>
<th>( r_3 )</th>
<th>( q_1 )</th>
<th>( q_2 )</th>
<th>( q_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2 1 0 0 1 3</td>
<td>4</td>
<td>1</td>
<td>1</td>
<td>1 1 1 1 2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>0 1 0 3 1</td>
<td>1 1 2 0 1 0 3</td>
<td>1</td>
<td>1</td>
<td>0 2 1 3 0 1</td>
<td>1</td>
<td>1</td>
<td>0 1 2 3 1 0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1 1 1 2 1 1</td>
<td>1 0 2 1 3 0 1</td>
<td>1</td>
<td>0 1 2 3 1 0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( n = 4 )</th>
<th>( a_{i_1i_2i_3} )</th>
<th>( i_1 )</th>
<th>( i_2 )</th>
<th>( i_3 )</th>
<th>( b_{h_1h_2h_3} )</th>
<th>( h_1 )</th>
<th>( h_2 )</th>
<th>( h_3 )</th>
<th>( s_1 )</th>
<th>( s_2 )</th>
<th>( s_3 )</th>
<th>( c_{r_1r_2r_3} )</th>
<th>( r_1 )</th>
<th>( r_2 )</th>
<th>( r_3 )</th>
<th>( q_1 )</th>
<th>( q_2 )</th>
<th>( q_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2 1 1 1 2 2</td>
<td>4 2 1 1 0 3 2</td>
<td>4</td>
<td>2 1 1 0 2 3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>2 0 1 0 4</td>
<td>1 2 2 0 0 1 4</td>
<td>1</td>
<td>2 0 2 0 4 1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1 2 1 3 0 2</td>
<td>14 1 2 1 2 1 2</td>
<td>14</td>
<td>1 1 2 2 2 1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>2 0 2 1 4 0</td>
<td>4 1 1 2 2 3 0</td>
<td>4</td>
<td>1 2 1 2 0 3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>1 1 2 3 2 0</td>
<td>1 0 2 2 4 1 0</td>
<td>1</td>
<td>0 2 2 4 0 1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
The following identities and symmetries are valid:

**Theorem II.**

\[ a_{i_1 i_2 i_3} = b_{i_3 i_1 i_2} = c_{i_2 i_3 i_1}, \]
\[ a_{i_1 i_2 i_3} = a_{i_1 i_3 i_2}, \]

and therefore,

\[ b_{i_3 i_1 i_2} = b_{i_2 i_1 i_3}, \]
\[ c_{i_2 i_3 i_1} = c_{i_3 i_2 i_1}. \]

Theorems III/1, III/2 and III/3 show relations between the randomization distribution and the elements \( a_{i_1 i_2 i_3}, b_{h_1 h_2 h_3} \) and \( c_{r_1 r_2 r_3}. \)

**Theorem III/1.** The sum of all elements \( a_{i_1 i_2 i_3} \) for a given \( n \) is equal to \( n! \).

Example: \( n = 4 \).

\[ a_{012} + a_{120} + a_{201} + a_{210} + a_{220} = 4 + 4 + 1 + 1 + 1 = 24 = 4! \]

**Theorem III/2.** For a given \( n = 2 \), the following relation is valid:

\[ \sum_{i_3=0}^{J_2} a_{i_1 i_2 i_3} = RU_{i_2}, \quad i_2 = 0, 1, \ldots, J_2, \]

where \( RU_{i_2} \) is the number of permutations of \( n \) natural numbers with \( j_2 \) runs up. Tables of the numbers \( RU_{i_2} \) are given in [3, p. 260, Table 7.2.2].

Example: \( n = 3 \).

\[ a_{201} = 1 = RU_0, \quad a_{210} + a_{111} = 1 + 4 = 5 = RU_1. \]

<table>
<thead>
<tr>
<th>Permutations</th>
<th>One run up (underlined)</th>
<th>Zero run up (underlined)</th>
</tr>
</thead>
<tbody>
<tr>
<td>123</td>
<td>123</td>
<td>123</td>
</tr>
<tr>
<td>132</td>
<td>132</td>
<td>132</td>
</tr>
<tr>
<td>231</td>
<td>231</td>
<td>231</td>
</tr>
<tr>
<td>213</td>
<td>213</td>
<td>213</td>
</tr>
<tr>
<td>312</td>
<td>312</td>
<td>312</td>
</tr>
<tr>
<td>321</td>
<td>321</td>
<td>321</td>
</tr>
<tr>
<td>Total</td>
<td>6 = n!</td>
<td>5 = RU_1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1 = RU_0</td>
</tr>
</tbody>
</table>

**Theorem III/3.** For a given \( n \geq 2 \) the following relation is valid:

\[ \sum_{i_2=0}^{J_2} a_{i_1 i_2 i_3} = P_{j_1}, \quad j_1 = 1, 2, \ldots, J_1, \]

where \( P_{j_1} \) is the number of permutations of \( n \) natural numbers with \( j_1 - 1 \) peaks. The
numbers \( P_{i_1} \) are tabulated in [3, p. 261, Table 7.3].

Example: \( n = 3 \).

\[
\begin{align*}
  a_{111} &= 4 = P_1, & a_{201} + a_{210} &= 1 + 1 = 2 = P_2.
\end{align*}
\]

<table>
<thead>
<tr>
<th>Permutations</th>
<th>One peak</th>
<th>Zero peak</th>
</tr>
</thead>
<tbody>
<tr>
<td>123</td>
<td>123</td>
<td>123</td>
</tr>
<tr>
<td>132</td>
<td>132</td>
<td>132</td>
</tr>
<tr>
<td>231</td>
<td>231</td>
<td>231</td>
</tr>
<tr>
<td>213</td>
<td>213</td>
<td>213</td>
</tr>
<tr>
<td>312</td>
<td>312</td>
<td>312</td>
</tr>
<tr>
<td>321</td>
<td>321</td>
<td>321</td>
</tr>
</tbody>
</table>

Total \( 6 = n! \) \( 2 = P_2 \) \( 4 = P_1 \)

Similar results can be obtained for \( b_{r_1r_2r_3} \) and \( c_{r_1r_2r_3} \) using Theorem II.

These theorems can be proved by mathematical induction. Theorem III/1 follows from Theorem III/2 or Theorem III/3 since the number of permutations of \( n \) natural numbers is equal to \( n! \).

Table II, in the microfiche section attached to this issue, lists the elements \( a_{i_1i_2i_3} \) for \( n = 0, 1, 2, \ldots, 15 \).

Putting \( u_0 = 0 \) the explicit terms of the series for \( s_n(u, k), c_n(u, k) \) and \( d_n(u, k) \) read (Theorem I, Theorem II and Table II):

\[
\begin{align*}
  s_n(u, k) &= u - \left( 1 + k^2 \right) \frac{u^3}{3!} + \left( 1 + 14k^2 + k^4 \right) \frac{u^5}{5!} - \left( 1 + 135k^2 + 135k^4 + k^6 \right) \frac{u^7}{7!} \\
  &+ \left( 1 + 1228k^2 + 5478k^4 + 1228k^6 + k^8 \right) \frac{u^9}{9!} \\
  &- \left( 1 + 11069k^2 + 165826k^4 + 165826k^6 + 11069k^8 + k^{10} \right) \frac{u^{11}}{11!} \\
  &+ \left( 1 + 99642k^2 + 4494351k^4 + 13180268k^6 + 4494351k^8 \\
  &\quad + 99642k^{10} + k^{12} \right) \frac{u^{13}}{13!} \\
  &- \left( 1 + 896803k^2 + 116294673k^4 + 834687179k^6 + 834687179k^8 \\
  &\quad + 116294673k^{10} + 896803k^{12} + k^{14} \right) \frac{u^{15}}{15!} + \cdots
\end{align*}
\]

\[
= \sum_{n_0=1; (n_0 \text{ odd})}^{\infty} (-1)^{(n_0-1)/2} \left( \sum_{j_2=0}^{(n_0-1)/2} \binom{(n_0-1)/2}{j_2} \right) \frac{u^{n_0}}{n_0!}, \quad j_1 = \frac{n_0 + 1}{2}
\]

(terms for \( n_0 \leq 7 \) are given in [2]);
TAYLOR SERIES COEFFICIENTS OF THE JACOBIAN FUNCTIONS 147

\[ \text{cn}(u, k) = 1 - \frac{u^2}{2!} + (1 + 4k^2) \frac{u^4}{4!} - (1 + 4k^2 + 16k^4) \frac{u^6}{6!} \]

\[ + (1 + 408k^2 + 912k^4 + 64k^6) \frac{u^8}{8!} \]

\[- (1 + 3688k^2 + 30768k^4 + 15080k^6 + 256k^8) \frac{u^{10}}{10!} \]

\[ + (1 + 33212k^2 + 870640k^4 + 1538560k^6 + 259328k^8 + 1024k^{10}) \frac{u^{12}}{12!} \]

\[- (1 + 298932k^2 + 22945056k^4 + 106923008k^6 + 65008896k^8 \]

\[ + 4180992k^{10} + 4096k^{12} \frac{u^{14}}{14!} + \cdots \]

\[ = 1 + \sum_{n_e=2; (n_e \text{ even})}^{\infty} (-1)^{n_e/2} \left( \sum_{h_1=0}^{n_e/2-1} b_{h_1} h_1 k^{2h_1} \right) \frac{u^{n_e}}{n_e!}, \quad h_3 = n_e/2 \]

(terms for \( n_e \leq 8 \) are given in [2]);

\[ \text{dn}(u, k) = 1 - k^2 \frac{u^2}{2!} + (4 + k^2) k^2 \frac{u^4}{4!} - (16 + 44k^2 + k^4) k^2 \frac{u^6}{6!} \]

\[ + (64 + 912k^2 + 408k^4 + k^6) k^2 \frac{u^8}{8!} \]

\[- (256 + 15080k^2 + 30768k^4 + 3688k^6 + k^8) k^2 \frac{u^{10}}{10!} \]

\[ + (1024 + 259328k^2 + 1538560k^4 + 870640k^6 + 33212k^8 + k^{10}) \frac{u^{12}k^2}{12!} \]

\[- (4096 + 4180992k^2 + 65008896k^4 + 106923008k^6 \]

\[ + 22945056k^8 + 298932k^{10} + k^{12} \frac{u^{14}k^2}{14!} + \cdots \]

\[ = 1 + \sum_{n_e=2; (n_e \text{ even})}^{\infty} (-1)^{n_e/2} \left( \sum_{r_3=1}^{n_e/2} c_{r_1} r_1^2 r_3^2 (r_3-1) \right) \frac{u^{n_e}k^2}{n_e!}, \quad r_2 = n_e/2 \]

(terms for \( n_e \leq 8 \) are given in [2]).

CEN

Saclay, France

