A Quadratically Convergent Iteration Method for Computing Zeros of Operators Satisfying Autonomous Differential Equations

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Abstract. If the Fréchet derivative $P'$ of the operator $P$ in a Banach space $X$ is Lipschitz continuous, satisfies an autonomous differential equation $P'(x) = f(P(x))$, and $f(0)$ has the bounded inverse $\Gamma$, then the iteration process
\[ x_{n+1} = x_n - \Gamma P(x_n), \quad n = 0, 1, 2, \ldots, \]
is shown to be locally quadratically convergent to solutions $x = x^*$ of the equation $P(x) = 0$. If $f$ is Lipschitz continuous and $\Gamma$ exists, then the global existence of $x^*$ is shown to follow if $P(x)$ is uniformly bounded by a sufficiently small constant. The replacement of the uniform boundedness of $P$ by Lipschitz continuity gives a semilocal theorem for the existence of $x^*$ and the quadratic convergence of the sequence \( \{x_n\} \) to $x^*$.

Successive approximations $x_1, x_2, \ldots$ to a solution $x = x^*$ of the operator equation $P(x) = 0$ in a Banach space $X$ can be obtained under suitable conditions from an iteration process of the form
\[ x_{n+1} = x_n - [P'(y_n)]^{-1} P(x_n), \quad n = 0, 1, 2, \ldots, \]
where the initial approximation $x_0$ and the sequence $\{y_n\}$ are given, and the existence of the inverses of the (Fréchet) derivatives $\{P'(y_n)\}$ and the convergence of the sequence $\{x_n\}$ to $x^*$ can be guaranteed. Special cases of (1) are Newton's method ($y_n = x_n$) and the modified Newton's method ($y_n = x_0$); so methods of this type may be characterized as variants of Newton's method, or Newton-like methods ([2], [3]).

1. Local Convergence. It will be assumed that $P(x^*) = 0$ and $\|P'(x) - P'(y)\| \leq K\|x - y\|$, at least in a sufficiently large region containing $x^*$. The inequality [4]
\[ \|x_{n+1} - x^*\| \leq \frac{1}{2} K\|P'(y_n)\|^{-1} \|x_n - x_n - y_n\| + \|y_n - x^*\|\|x_n - x^*\| \]
is useful for estimating the rate of convergence of $\{x_n\}$ to $x^*$. If one takes $y_n = \lambda_n x_n + (1 - \lambda_n)x^*$, $0 \leq \lambda_n \leq 1$, then $\|x_n - y_n\| + \|y_n - x^*\| = \|x_n - x^*\|$, and one has
\[ \|x_{n+1} - x^*\| \leq \frac{1}{2} K\|P'(y_n)\|^{-1} \|x_n - x^*\|, \]
which shows that convergence will be quadratic if the inverses $[P'(y_n)]^{-1}$ are uniformly

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bounded. The method of present interest is obtained by taking \( \lambda_n = 0 \), so that \( y_n = x^* \). If now \( \Gamma = [P'(x^*)]^{-1} \) exists and \( \|\Gamma\| \leq B^* \), then the iteration process

\[
x_{n+1} = x_n - \Gamma P(x_n), \quad n = 0, 1, 2, \ldots,
\]

will be quadratically convergent, with

\[
\|x_{n+1} - x^*\| \leq \frac{1}{2} KB^* \|x_n - x^*\|^2.
\]

The iteration process (4) has the advantages of the quadratic convergence of Newton's method and the simplicity of the modified Newton's method, as the operator \( \Gamma \) is calculated once and for all. This method can be realized for operators \( P \) which satisfy an autonomous differential equation

\[
P'(x) = f(P(x)),
\]

as \( P'(x^*) = f(0) \) can be evaluated without knowing the value of \( x^* \). With the above assumptions one has the following result.

**Theorem 1.** If \( \Gamma = [f(0)]^{-1} \) exists, \( \|\Gamma\| \leq B^* \), and \( x_0 \) is such that

\[
\theta = \frac{\frac{1}{2} KB^* \|x_0 - x^*\|}{1 + \frac{1}{2} KB^* \|x_0 - x^*\|} < 1,
\]

then the sequence \( \{x_n\} \) defined by (4) converges to \( x^* \), with

\[
\|x_n - x^*\| \leq \theta^{2n-1} \|x_0 - x^*\|, \quad n = 1, 2, \ldots.
\]

**Proof.** Inequality (8) follows from (5) and (7) by mathematical induction.

For example, the iteration process

\[
x_{n+1} = x_n - \frac{1}{N} (e^{x_n} - N), \quad n = 0, 1, 2, \ldots,
\]

converges quadratically to the solution \( x^* = \ln N \) of \( P(x) = e^x - N = 0 \) for sufficiently close initial approximations \( x_0 \); here (6) is \( P'(x) = P(x) + N \).

2. A Global Existence Theorem. It will be assumed that \( \Gamma = [f(0)]^{-1} \) exists, \( \|\Gamma\| \leq B^* \), and conditions for the existence of \( x^* \) will be obtained.

**Theorem 2.** If \( f \) is Lipschitz continuous with constant \( \alpha \), \( \|P(x)\| \leq \beta \), and

\[
\rho = \frac{\alpha \beta B^*}{1 - \alpha} < 1,
\]

then the equation \( P(x) = 0 \) has a unique solution \( x^* \) to which the sequence \( \{x_n\} \) defined by (4) converges, with

\[
\|x^* - x_n\| \leq \frac{\rho^n}{1 - \rho} \|x_1 - x_0\|, \quad n = 0, 1, 2, \ldots.
\]

**Proof.** The iteration process (4) may be written as \( x_{n+1} = \Gamma F(x_n) \), \( n = 0, 1, 2, \ldots \), where \( F(x) = f(0)x - P(x) \). From

\[
F'(x) = f(0) - P'(x) = f(0) - f(P(x))
\]

and the Lipschitz continuity of \( f \), it follows that

\[
\|F'(x)\| \leq \alpha \|P(x)\|,
\]

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and the theorem follows from (10) and the contraction mapping principle [3].

If $P'$ is Lipschitz continuous in a neighborhood of $x^*$, then the convergence of the sequence $\{x_n\}$ will be quadratic within this neighborhood as soon as inequality (7) holds with $x_0$ replaced by an iterate $x_n$ sufficiently close to $x^*$.

3. A Semilocal Existence Theorem. If $f$ and $P$ are Lipschitz continuous with constants $\alpha$ and $\gamma$, respectively, then it follows from (6) that $P'$ is Lipschitz continuous with constant $K = \alpha \gamma$. Furthermore,

\begin{equation}
\|P(x)\| \leq \|P(x_0)\| + \gamma \|x - x_0\|.
\end{equation}

For $r = \|x - x_0\|$, define

\begin{equation}
\rho(r) = \alpha B^* \|P(x_0)\| + B^* K r.
\end{equation}

If $\rho(0) = \alpha B^* \|P(x_0)\| < 1$, then inequality (10) and the contraction mapping principle [3, pp. 84—85] give the following result.

Theorem 3. If

\begin{equation}
\Delta = (1 - \alpha B^* \|P(x_0)\|)^2 - 4 B^* K \|x_1 - x_0\| \geq 0,
\end{equation}

then a solution $x^*$ of the equation $P(x) = 0$ exists in the closed ball

\begin{equation}
V = \left\{ x : \|x - x_0\| \leq \frac{1 - \alpha B^* \|P(x_0)\| - \sqrt{\Delta}}{2 B^* K} = r^* \right\},
\end{equation}

and is unique in the open ball

\begin{equation}
U = \left\{ x : \|x - x_0\| < \frac{1 - \alpha B^* \|P(x_0)\|}{B^* K} \right\}.
\end{equation}

By itself, the contraction mapping principle only guarantees that

\begin{equation}
\|x_n - x^*\| \leq (\rho^*)^r r^*, \quad n = 0, 1, 2, \ldots,
\end{equation}

where

\begin{equation}
\rho^* = \rho(r^*) = \frac{1}{2}(1 + \alpha B^* \|P(x_0)\| - \sqrt{\Delta}).
\end{equation}

By Theorem 1, however, the convergence of the sequence $\{x_n\}$ to $x^*$ will be quadratic for $n = N, N + 1, \ldots$, where $N$ is the smallest nonnegative integer satisfying the inequality

\begin{equation}
\theta = \frac{1}{2} K B^* (\rho^*)^N r^* < 1.
\end{equation}