A Necessary Condition for $A$-Stability of Multistep Multiderivative Methods

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Abstract. The region of absolute stability of multistep multiderivative methods is studied in a neighborhood of the origin. This leads to a necessary condition for $A$-stability. For methods where $\rho(\zeta)/\zeta$ has no roots of modulus 1 this condition can be checked very easily. For Hermite interpolatory and Adams type methods a necessary condition for $A$-stability is found which depends only on the error order and the number of derivatives used at $(x_{n+k}, y_{n+k})$.

1. Introduction and Results. A multistep method using higher derivatives for solving the initial value problem $y' = f(x, y), y(a) = \eta$ is given by

$$ \sum_{i=0}^{k} \alpha_i y_{n+i} - \sum_{j=1}^{l} h^j \sum_{i=0}^{k} \beta_{ji} f^{(j)}(x_{n+i}, y_{n+i}) = 0, \quad n = 0, 1, 2, \ldots. $$

Here, $\alpha_i, \beta_{ji}$ are real constants, $\alpha_k \neq 0$, $\sum_{i=0}^{k} |\beta_{ji}| \neq 0$, $|\alpha_0| + \sum_{j=1}^{l} |\beta_{j0}| \neq 0$, $x_n = a + nh$, $h > 0$, and

$$ f^{(1)}(x, y) = f(x, y); $$

$$ f^{(j+1)}(x, y) = \frac{\partial f^{(j)}(x, y)}{\partial x} + f(x, y) \frac{\partial f^{(j)}(x, y)}{\partial y}, \quad j = 1, 2, \ldots, l - 1. $$

It is well known that the method has order $p$ if

$$ \rho(e^2) - \sum_{j=1}^{l} z^j \sigma_j(e^2) = \sum_{j=p+1}^{\infty} C_j z^j, \quad C_{p+1} \neq 0, $$

where $\rho(\zeta)$ and $\sigma_j(\zeta)$ are the polynomials

$$ \rho(\zeta) = \sum_{i=0}^{k} \alpha_i \zeta^i, \quad \sigma_j(\zeta) = \sum_{i=0}^{k} \beta_{ji} \zeta^i, \quad j = 1, 2, \ldots, l. $$

We shall always assume that the polynomials $\rho$ and $\sigma_j, j = 1, 2, \ldots, l$, have no common factor. The method is convergent if and only if $p \geq 1$ and $\rho(\zeta)$ is a simple von Neumann polynomial; that is, if $\zeta$ is a root of $\rho(\zeta)$, then $|\zeta| < 1$; and if $|\zeta| = 1$, then it is a simple root (see R. Jeltsch [8]).

If the multistep method (1) is applied to the test equation $y' = \mu y, y(0) = 1, \mu$ complex, then (1) is a linear recurrence relation with constant coefficients. The corresponding characteristic equation is

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For each \( z \), (3) has \( k \) roots \( \zeta_i(z) \), \( i = 1, 2, \ldots, k \). The set \( A = \{ z \mid |\zeta_i(z)| < 1, i = 1, 2, \ldots, k \} \) is called the region of absolute stability. Let \( \partial A = \bar{A} - A \), where \( \bar{A} \) is the closure of \( A \). A method is called \( A \)-stable if \( A \) contains the whole left-hand plane \( \Re z < 0 \).

In several articles the boundary \( \partial A \) of \( A \) has been plotted in order to determine whether a method is \( A \)-stable or not, see Brown [1], Enright [4], Jeltsch [7]. However, if all growth parameters \( \lambda_j \), given by (4), are positive, then \( \partial A \) will be extremely close to the imaginary axis for \( z \) close to 0. Roundoff errors may defeat the attempt to determine whether \( \partial A \) is in a neighborhood of \( z = 0 \) in \( H^+ = \{ z \in \mathbb{C} \mid \Re z > 0 \} \) or in \( H^- = \{ z \in \mathbb{C} \mid \Re z < 0 \} \). Our results fill this gap. In particular, we shall find a necessary condition for \( A \)-stability. It should be noted that a method which violates this condition may still behave numerically almost like an \( A \)-stable method even though it is not \( A \)-stable. In Section 2 this necessary condition for \( A \)-stability is applied to Hermite interpolatory and Adams-type multistep multiderivative methods; and it is found that these cannot be \( A \)-stable if the error order \( p \) is equal to \( 2l_k + 1 \) modulo 4, where

\[
l_k = \begin{cases} 
0 & \text{if } \sum_{j=1}^{l} |\beta_{jk}| = 0, \\
t & \text{if } \sum_{j=t+1}^{l} |\beta_{jk}| = 0 \text{ and } \beta_{tk} \neq 0.
\end{cases}
\]

The proofs are given in Section 3.

Let \( \zeta_j, j = 1, 2, \ldots, s \), be the roots of \( \rho(\zeta) \) with modulus 1. Let us introduce the growth parameters

(4) \[ \lambda_j = \frac{\sigma_1(\zeta_j)}{\zeta_j \rho'(\zeta_j)}, \quad j = 1, 2, \ldots, s, \]

and

(5) \[ \mu_j = \frac{1}{\zeta_j \rho'(\zeta_j)} \left( \sigma_2(\zeta_j) + \zeta_j \lambda_j \sigma_1'(\zeta_j) - \frac{1}{2} \zeta_j^2 \lambda_j^2 \rho''(\zeta_j) \right), \quad j = 1, 2, \ldots, s. \]

Furthermore, let the method have order \( p \geq 1 \). Then we define recursively

(6) \[ c_j = \left( C_j - \sum_{i=p+1}^{j} c_{j-i} \right)/s_0, \quad j = p + 1, p + 2, \ldots, 2p, \]

where \( s_0, s_1, \ldots, s_{p-1} \) are given by

(7) \[ \sum_{j=1}^{l} jz^{j-1} \sigma_j(z^2) = \sum_{i=0}^{p-1} s_i z^i + O(z^p). \]

For the disk \( \{ z \in \mathbb{C} \mid |z| < R \} \) we shall use the symbol \( D(R) \).

**Theorem 1.** Let the multistep method of form (1) be convergent, of order \( p \geq 1 \) and let \( \rho(\zeta) \) have \( s \) roots of modulus 1, \( \zeta_i, i = 1, 2, \ldots, s \), with \( \zeta_1 = 1 \). Let \( \lambda_i \) be real and positive, \( i = 1, 2, \ldots, s \), and define
where $\lambda_j$ and $\mu_j$ are given by (4) and (5), respectively. Assume that one of the conditions (I), (II$_1$)–(II$_4$) holds, where

(I) $\delta < 0$.

(II$_1$) $\delta > 0$, $p$ odd, $c_{p+1} (-1)^{(p+1)/2} > 0$.

(II$_2$) $\delta > 0$, $p$ even, $c_{p+2q} (-1)^{(p/2)+q} > 0$, $c_{p+2j} = 0$, $j = 1, 2, \ldots, q-1$,

for some $q \leq p/2$.

(II$_3$) $\delta > 0$, $p$ odd, $c_{p+1} (-1)^{(p+1)/2} < 0$.

The numbers $c_j$, $j = p + 1, p + 2, \ldots, 2p$, are given by (6). Then there exists a disk $D = D(R), R > 0$, such that $\gamma = \partial A \cap D$ is a continuously differentiable curve which intersects the real axis and the imaginary axis only at $z = 0$. The imaginary axis is tangent to $\gamma$ at $z = 0$. $\gamma$ divides $D$ in two simply connected regions $D^- = A \cap D$ and $D^+ = D - D^-$, see Figure 1. Moreover, each of the conditions (I), (II$_3$), (II$_4$) implies that $D^- \subset \mathbb{H}^-$ while each of the conditions (II$_1$), (II$_2$) implies that $D^+ - \{0\} \subset \mathbb{H}^+$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Absolute stability region in a neighborhood of the origin}
\end{figure}

(a) if one of the conditions (I), (II$_3$), (II$_4$) holds

(b) if one of the conditions (II$_1$), (II$_2$) holds

Remarks. 1. Using (2) and (7), one finds the explicit formulas

\[
C_n = \frac{1}{n!} \sum_{m=0}^{k} \alpha_m m^n - \sum_{j=1}^{\min\{n, k\}} \frac{1}{(n-j)!} \sum_{m=0}^{k} \beta_{jm} m^{n-j},
\]

\[
\min\{n+1, k\} \leq j \leq \min\{n, k\}
\]

and

\[
s_n = \sum_{j=1}^{\min\{n+1, k\}} \frac{j}{(n+1-j)!} \\sum_{m=0}^{k} \beta_{jm} m^{n+1-j}, \quad n = 0, 1, 2, \ldots, p - 1.
\]
Moreover, from (6), (7) and (2) follows

\[ c_{p+1} = \frac{C_{p+1}}{\rho'(1)} \neq 0 \]

and

\[ c_{p+2} = \left( C_{p+2} - \frac{C_{p+1}}{\rho'(1)} \left( o_1'(1) + 2o_2'(1) \right) \right) / \rho'(1). \]

2. Let \( s = 1 \). If \( p \) is odd, then Theorem 1 describes \( \partial A \) close to \( z = 0 \) in all cases since \( c_{p+1} \neq 0 \). The methods with \( p \) even and \( c_{p+2j} = 0, j = 1, 2, \ldots, p/2 \), are not covered by Theorem 1. However, there are only a few methods with this property since one has the following result by Griepentrog [6]. There exists no \( k \)-step method of form (1) with \( k \geq 2 \) and \( s = 1 \) for which \( \partial A \) lies exactly on the imaginary axis in a neighborhood of \( z = 0 \). Moreover, a one-step method of form (1) with \( p \geq 1 \) has \( \partial A \) on the imaginary axis in a neighborhood of \( z = 0 \) if and only if \( \beta_{j_1} = (-1)^{j_1} \beta_{j_0}, j = 1, 2, \ldots, l \).

**THEOREM 2.** It is necessary for a method to be \( A \)-stable that all growth parameters are real and nonnegative, \( \delta \geq 0 \) and either (III) or (IV) holds, where \( \delta \) is defined as in Theorem 1 and

1. \( p \) odd, \( c_{p+1}(-1)(p+1)^{1/2} > 0 \).
2. \( p \) even, either \( c_{p+2j} = 0, j = 1, 2, \ldots, p/2 \), or \( c_{p+2j}(-1)(p/2)^{1/2} + q > 0 \), \( c_{p+2j} = 0, j = 1, 2, \ldots, q-1 \), for some \( q \leq p/2 \).

**Remark.** This necessary condition for \( A \)-stability is very easy to check for \( s = 1 \). If \( p \) is odd, only \( c_{p+1} \) has to be calculated. If \( p \) is even one finds for most methods that \( c_{p+2} \neq 0 \); and hence, only \( c_{p+2} \) has to be calculated. The following lemma simplifies the problem of determining the sign of \( c_{p+1} \).

**LEMMA.** Let the multistep method using higher derivatives be convergent, then

\[ \text{sign } \rho'(1) = \text{sign } \alpha_k. \]

**Proof.** Since the method is convergent, all roots of \( \rho(z) \) and \( \rho'(z) \) lie in the unit disk and hence the lemma holds.

2. **Application to Hermite Interpolatory and Adams Type Methods.**

**Definition 1.** A linear multistep method using higher derivatives of the form

\[ \sum_{i=0}^{k} \alpha_i y_{n+i} - \sum_{i=0}^{k} \sum_{j=1}^{l_i} h_i \beta_{i,j} f^{(j)}(x_{n+i}, y_{n+i}) = 0 \]

is called Hermite interpolatory if the error order \( p \) is at least \( \sum_{i=0}^{k} l_i + k - 1 \).

In Jeltsch [9] the following theorem is proved.

**THEOREM 3.** Let a set of nonnegative integers \( l_0, l_1, \ldots, l_k \) with \( \max_{i=0,1,\ldots,k} l_i = l > 0 \) be given. Then there exists a unique Hermite interpolatory multistep method with the given \( l_i, \alpha_k \neq 0 \) and \( \beta_{i,k} \neq 0 \). The error order is \( p = \sum_{i=0}^{k} l_i + k - 1 \) and one has

\[ \text{sign } C_{p+1} = (-1)^{l_k} \text{sign } \alpha_k. \]

A similar result can be established for Adams-type methods which are defined as follows.
Definition 2. A linear multistep method using higher derivatives is said to be of Adams type if it is of the form

\[ y_{n+k} - y_{n+k-1} - \sum_{i=0}^{k} \frac{h^i}{i!} f^{(i)}(x_{n+i}, y_{n+i}) = 0, \]

and its error order \( p \) is at least \( \sum_{i=0}^{k} l_i \).

Theorem 4. Let a set of nonnegative integers \( l_0, l_1, \ldots, l_k \) with \( \max l_i = l > 0 \) be given. Then there exists a unique Adams-type multistep method with the given \( l_i \) and \( \beta_{i+k} \neq 0 \). The error order is \( p = \sum_{i=0}^{k} l_i \) and one has

\[ \text{sign} C_{p+1} = (-1)^{l_k} \text{sign} \alpha_k. \]

Using Theorems 2, 3, 4 and the Lemma, one then finds immediately the

Theorem 5. A convergent linear multistep method using higher derivatives which is of Adams type or Hermite interpolatory cannot be \( A \)-stable if the error order \( p \) satisfies

\[ p = 2k + 1 \mod 4. \]

Example 1. Brown's methods are interpolatory with

\[ l_0 = l_1 = \cdots = l_{k-1} = 0, \quad l_k = l. \]

Hence, the methods are not \( A \)-stable if \( p = 2l + 1 \mod 4 \). Especially the methods with \( k = 4, l = 2, p = 5; k = 5, l = 3, p = 7 \) and \( k = 6, l = 4, p = 9 \) are not \( A \)-stable. The method with \( p = 10, k = 7 \) and \( l = 4 \) is not covered by Theorem 5. However, A. H. Sipilä has computed \( C_{11} \) and \( C_{12} \) using rational arithmetic and it was found that

\[ c_{12} = (C_{11}/3p'(1)) (-4.653007 \ldots). \]

Hence, by our Lemma and Theorem 3, one has \( c_{12}(-1)^{p/2+1} < 0 \). Hence, by Theorem 2 this method is not \( A \)-stable. Note that in Brown [1] the plots of \( \partial A \) lead to the wrong conclusion that these methods are \( A \)-stable.

Example 2. Consider the linear one-step methods using higher derivatives which are based on the \((r, l)\) entry of the Padé table of \( \exp(x) \), see Jeltsch [8] or Ehle [3, p. 89]. These methods have order \( p = r + l \) and are interpolatory. It is known, see Ehle [3], that the methods are \( A \)-stable for \( r = l, l-1, l-2 \). From Theorem 5 it follows that the methods are not \( A \)-stable for \( r = l-3 \). This result has been found by Ehle [3].

Example 3. Enright's second derivative methods are of Adams type with \( l_0 = l_1 = \cdots = l_{k-1} = 1 \) and \( l_k = 2 \), with order \( p = k + 2 \), see Enright [4]. Using the Lemma and Theorems 1 and 4, one finds that for \( k = 3 \mod 4 \) the region of absolute stability behaves at the origin as given in Figure 1a and for \( k = 5 \mod 4 \) as given in Figure 1b.

3. Proof of the Results.

Proof of Theorem 1. The algebraic function \( \xi(z) \) which satisfies (3) has \( k \) branches \( \xi_j(z) \) with \( \xi_j(0) = \xi_j, j = 1, 2, \ldots, k \). Since \( |\xi_j(z)| < 1 \) for \( j = s + 1, s + 2, \ldots, k \) there exists a \( D(R_1) \), \( R_1 > 0 \) such that \( |\xi_j(z)| < 1 \) for all \( z \in D(R_1) \),
\[ j = s + 1, s + 2, \ldots, k. \quad \xi_j(0), j = 1, 2, \ldots, s, \] are simple zeros of \( \rho(\xi) \); and hence, there exists a disk \( D(R_2), 0 < R_2 < R_1, \) such that the branches \( \xi_j(z) \) are analytic in \( D(R_2). \) By the method of undetermined coefficients one finds
\begin{equation}
(12) \quad \xi_j(z) = \xi_j(0)(1 + \lambda_j z + \mu_j z^2 + O(z^3)), \quad j = 1, 2, \ldots, s;
\end{equation}
and hence,
\begin{equation}
(13) \quad \frac{d\xi_j(z)}{dz} \bigg|_{z=0} = \xi_j(0)\lambda_j \neq 0, \quad j = 1, 2, \ldots, s.
\end{equation}

Hence, there exists an \( R_3, 0 < R_3 < R_2, \) such that the mapping \( \xi_j(z): z \to \xi = \xi_j(z) \) is one to one on \( z \in D(R_3). \) Moreover, \( R_3 \) can be chosen so small that the curves \( \gamma^{(l)} = \{z \in D(R_3) \mid |\xi_j(z)| = 1\} \) are continuously differentiable. Clearly, \( \{0\} \in \gamma^{(l)} \) and from (13) it follows that the imaginary axis is tangent to \( \gamma^{(l)} \) at \( z = 0. \) If \( i \neq j, \) then either \( \gamma^{(l)} \cap \gamma^{(l)} \) is a finite set or \( \gamma^{(l)} \cap \gamma^{(l)} \) is a continuous curve which contains \( z = 0. \) Hence, there exists \( \tilde{R}, 0 < \tilde{R} < R_3, \) such that either \( \gamma_j \equiv \gamma_i \) and \( \gamma_j \subseteq [\tilde{R}, \tilde{R}] = \{0\} \) for \( j = 1, 2, \ldots, s, \) where \( \gamma_j = D(\tilde{R}) \cap \gamma^{(l)}. \) Each \( \gamma_j \) separates \( D(\tilde{R}) \) in the two sets \( D_j^- = \{z \in D(\tilde{R}) \mid |\xi_j(z)| < 1\} \) and \( D_j^+ = \{z \in D(\tilde{R}) \mid |\xi_j(z)| > 1\}. \) Clearly, \( (-\tilde{R}, 0) \subseteq D_j^- \), \( j = 1, 2, \ldots, s. \) We distinguish now two cases:

(i) Consider \( \xi_j(z), j = 2, 3, \ldots, s. \) With \( z = iy, y \in (-\tilde{R}, \tilde{R}), \) one finds from (12)
\begin{equation}
(14) \quad |\xi_j(iy)| = |1 - y^2 \text{Re}\mu_j + i(\lambda_j y - y^2 \text{Im}\mu_j) + O(y^3)|
\end{equation}
\begin{equation}
= \sqrt{1 - y^2(2\text{Re}\mu_j - \lambda_j^2) + O(y^3))}.
\end{equation}

(ii) Consider \( \xi_1(z). \) It is well known, see, e.g. Gear [5] that \( \xi_1(z) - e^z = O(z^{p+1}). \) Since \( \xi_1(z) \) is analytic at the origin, we can write
\begin{equation}
(15) \quad \xi_1(z) = e^z \left( 1 - \sum_{j=p+1}^{2p} c_j z^j + O(z^{2p+1}) \right).
\end{equation}

If one substitutes (15) in (3) and uses (2), one finds easily that \( c_j, j = p + 1, p + 2, \ldots, 2p, \) are determined by (6) and (7). Note that \( c_j \) is a real number. Let \( p \) be odd. Then \( c_{p+1}i^{p+1} = c_{p+1}(-1)(p+1)/2 \) is real and nonzero. Hence we find for \( z = iy, y \) real,
\begin{equation}
(16) \quad |\xi_1(iy)| = |e^{iy}| |1 - c_{p+1}i^{p+1}y^{p+1} + O(y^{p+2})|
\end{equation}
\begin{equation}
= \sqrt{1 - 2c_{p+1}(-1)(p+1)/2 y^{p+1} + O(y^{p+2})} \quad \text{for } p \text{ odd.}
\end{equation}

Let \( p \) be even. Then \( c_{p+2j}i^{p+2j} = c_{p+2j}(-1)(p/2)+j \) is real for \( j = 1, 2, \ldots, p/2. \) Hence, we find for \( z = iy, y \) real,
\begin{equation}
(17) \quad |\xi_1(iy)| = |e^{iy}| \left| 1 - \sum_{j=1}^{p/2} c_{p+2j}i^{p+2j}y^{p+2j} \right|
\end{equation}
\begin{equation}
= \sqrt{1 - 2 \sum_{j=0}^{(p/2)-1} c_{p+2j+1}i^{p+2j+1}y^{p+2j+1} + O(y^{2p+1})} \quad \text{for } p \text{ even.}
\end{equation}
Assume now that condition (I) holds. Then it follows from (14) that there exists \( R, 0 < R < \tilde{R} \), such that \( |\xi_j(i\psi)| > 1 \) for \( y \) real, \( 0 < |y| < R \) for at least one \( j \in \{2, 3, \ldots, s\} \). Therefore, \( D^- = A \cap D(R) = \bigcap_{j=1}^s D_j^+ \cap D(R) \subset H^- \).

If (II_1), (II_2), respectively, hold then by (16), (17) and (14) there exists \( R, 0 < R < \tilde{R} \), such that \( |\xi_j(i\psi)| < 1 \) and \( |\xi_j(i\psi)| < 1, j = 2, 3, \ldots, s \), for \( y \) real, \( 0 < |y| < R \). Therefore, \( D^+ = \bigcup_{j=1}^s D_j^+ \cap D(R) \) satisfies \( D^+ - \{0\} \subset H^+ \) since \( D_j^+ \cap D(R) - \{0\} \subset H^+ \). Similarly, one finds that (II_3), (II_4) imply \( D^- \subset H^- \). This completes the proof of Theorem 1.

Proof of Theorem 2. Let \( \lambda_j = de^{i\phi}, d > 0, \phi \in (0, 2\pi) \). Clearly,

\[
\psi = \frac{3\pi}{2} - \frac{\phi}{2} \in \left( \frac{\pi}{2}, \frac{3\pi}{2} \right) \quad \text{and} \quad \psi + \phi \in \left( \frac{3\pi}{2}, \frac{5\pi}{2} \right).
\]

Hence, using (12), one finds

\[
|\xi_j(re^{i\psi})| = |1 + rde^{i(\phi + \psi)} + O(r^2)| > 1
\]

for all \( r > 0, r \) sufficiently small. Therefore, the method is not \( A \)-stable. Let \( \lambda_j \geq 0, j = 1, 2, \ldots, s, \) and \( \delta < 0 \). From (14) follows immediately that the method is not \( A \)-stable. Similarly, using (16) and (17) one finds that (III), (IV) are necessary for \( A \)-stability. This establishes Theorem 2.

Proof of Theorem 4. In Jeltsch [9] it is shown that to given nonnegative integers \( l_0, l_1, \ldots, l_k \) with max \( l_i = l > 0 \) there exists a unique Adams-type method with the given \( l_i, \beta_{jk} \neq 0 \) and that the error order \( p = \Sigma_{i=0}^k l_i \). Hence, it remains to show that

\[
\text{sign} C_{p+1} = (-1)^l \text{sign} \alpha_k.
\]

To show this we construct the method explicitly. Let \( P(x) \) be the interpolation polynomial of degree \( \Sigma_{i=0}^k l_i - 1 \) which satisfies

\[
P^{(j-1)}(x_i) = y^{(j)}(x_{n+i}, y_{n+i}), \quad j = 1, 2, \ldots, l_i, i = 0, 1, 2, \ldots, k.
\]

The multistep method is obtained by setting

\[
y_{n+k} - y_{n+k-1} = \int_{x_{n+k-1}}^{x_{n+k}} P(x) \, dx.
\]

To find the error order and \( C_{p+1} \) we apply the method given by (19) to a sufficiently smooth function \( y(x) \). Clearly,

\[
y'(x) - P(x) = f^*(x) \prod_{i=0}^k (x - x_{n+i})^{l_i},
\]

where \( f^*(x) \) is the generalized divided difference of the function \( y'(x) \) on the set

\[
S = \{x, x_n, x_{n+1}, \ldots, x_n, x_{n+1}, \ldots, x_{n+1}, x_{n+1}, \ldots, x_{n+k}, x_{n+k}, \ldots, x_{n+k}\},
\]

see e.g. Conte and de Boor [2, p. 223]. Hence,
\[
\int_{x_{n+k-1}}^{x_{n+k}} (y'(x) - P(x)) \, dx = \int_{x_{n+k-1}}^{x_{n+k}} f^*(x) \prod_{i=0}^{k} (x - x_{n+i})^{l_i} \, dx
\]

\[
= f^*(z) \int_{x_{n+k-1}}^{x_{n+k}} \prod_{i=0}^{k} (x - x_{n+i})^{l_i} \, dx,
\]

since the factor \( \prod_{i=0}^{k} (x - x_{n+i})^{l_i} \) does not change sign in the interval \([x_{n+k-1}, x_{n+k}]\); and hence, the second mean value theorem of the integral calculus can be applied, \( z \in [x_{n+k-1}, x_{n+k}] \). But \( f^*(z) = 1/((p+1)!) (x^{p+1}(\eta), \) where \( \eta \in [x_n, x_{n+k}] \); and hence,

(21) \[
\int_{x_{n+k-1}}^{x_{n+k}} (y'(x) - P(x)) \, dx = Kh^{p+1} y^{(p+1)}(\eta),
\]

where

(22) \[
K = \frac{1}{(p+1)!} \int_{0}^{1} \prod_{i=0}^{k} (s + k - 1 - i)^{l_i} \, ds.
\]

Using (21), it is easy to see that the method given by (19) is of error order \( p \) and that \( C_{p+1} = K \). From (22) follows that \( \text{sign} \, C_{p+1} = (-1)^k \). The proof of Theorem 4 is complete since there exists exactly one Adams-type method.

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