Growth of Partial Sums of Divergent Series

By R. P. Boas, Jr.

Abstract. Let $\Sigma f(n)$ be a divergent series of decreasing positive terms, with partial sums $s_n$, where $f$ decreases sufficiently smoothly; let $\varphi(x) = \int_1^x f(t) dt$ and let $\psi$ be the inverse of $\varphi$. Let $n_A$ be the smallest integer $n$ such that $s_n \geq A$ but $s_{n-1} < A$ ($A = 2, 3, \ldots$); let $\gamma = \lim \{\Sigma_{k=1}^n f(k) - \varphi(n)\}$ be the analog of Euler’s constant; let $m = \lfloor \psi(A - \gamma) \rfloor$. Call $\omega$ a Comtet function for $\Sigma f(n)$ if $n_A = m$ when the fractional part of $\psi(A - \gamma)$ is less than $\omega(A)$ and $n_A = m + 1$ when the fractional part of $\psi(A - \gamma)$ is greater than $\omega(A)$. It has been conjectured that $\omega(A) = 1/2$ is a Comtet function for $\Sigma 1/n$. It is shown that in general there is a Comtet function of the form

$$\omega(A) = \frac{1}{2} + \frac{1}{24} \left[ \left\lfloor \frac{1}{f'(m)} / f(m) \right\rfloor \right] (1 + o(1)).$$

For $\Sigma 1/n$ there is a Comtet function of the form $1/2 + 1/(24m) - 1/(48m^2)(1 + o(1))$. Some numerical results are presented.

1. Introduction. If $\Sigma_{n=1}^\infty f(n)$ is a divergent series of positive terms that approach 0, one can measure how fast it diverges by seeing how fast the partial sums $s_n$ increase. Numerical data for representative series are given in the appendix to [4] (p. 69), but some of them are rather inaccurate. The present note grew out of an attempt to recompute this table. The results are given in the table on p. 259; they correct some of the entries in [4] and give a few more. The entries less than $10^6$ were found by direct machine evaluation of the partial sums; most of these were checked, and the other entries were obtained, by using Theorem 2 below, which is a generalization of known results for the harmonic series [2], [3]. The entries for the harmonic series (no. 4 in the table) were originally calculated by Wrench and published in [2].

A classical theorem of Maclaurin and Cauchy (see [4, p. 45]) states that if $f$ is positive and decreases to 0, then $s_n - \int_1^n f(t) dt$ approaches a limit. When $f(n) = 1/n$, this limit is Euler’s constant $\gamma$; I use the same notation in the general case. The table includes approximations to $\gamma$ for each series.

Notation. $f$ is a positive decreasing function with $f(\infty) = 0$, such that, at least for $n = 1, 2, 3, |f^{(k)}(x)|$ decreases for large $x$ and is $O(f(x)x^{-n})$, and with $\Sigma f(n)$ divergent. We define $\varphi(x) = \int_1^x f(t) dt$; $\psi(y)$ is the inverse of $y = \varphi(x)$; we assume that $\psi''$ is eventually monotonic. Let $s_n = \Sigma_{k=1}^n f(k)$ and $\gamma = \lim_{n \to \infty} (s_n - \varphi(n))$. When $A$ is a positive integer, $n_A$ denotes the smallest integer $n$ such that $s_n \geq A$ but $s_{n-1} < A$.

Received February 2, 1976.

For functions $f$ satisfying these hypotheses, the existence of $\gamma$ suggests that $\psi(A - \gamma)$ ought to be a good estimate of $n_A$.

**Theorem 1.** For sufficiently large $A$, the number $n_A$ is one of the two integers closest to $\psi(A - \gamma)$.

Theorem 1 (with “sufficiently large” meaning “at least 2”) was proved for the harmonic series by Comtet [3]; this seems to have been the first really precise result in this direction.

Because of Theorem 1, $n_A$ is either $\lfloor \psi(A - \gamma) \rfloor$ or $\lceil \psi(A - \gamma) \rceil + 1$. Let us introduce a function $\omega$ such that the first case occurs when the fractional part of $\psi(A - \gamma)$ is less than $\omega(A)$; the second, when the fractional part of $\psi(A - \gamma)$ is greater than $\omega(A)$. Of course, $\omega$ is not uniquely determined. I propose to call such a function a Comtet function for $f$ (or for $\Sigma_f(n)$).

It has been conjectured that $\omega(A) = \frac{1}{2}$ is a Comtet function for the harmonic series, and proved [2] that this series has a Comtet function of the form $\omega(A) = \frac{1}{2} + O(e^{-A})$.

**Theorem 2.** Every series of the form $\Sigma f(n)$ (with the hypotheses stated above) has a Comtet function of the form

$$
\omega(A) = \frac{1}{2} + \frac{1}{24} (|f'(m)|/f(m))(1 + o(1)),$$

where $m = \lfloor \psi(A - \gamma) \rfloor$.

For any specific $f$ we can improve Theorem 2 by more detailed calculation. We shall do this for the harmonic series.

**Theorem 3.** For $\Sigma 1/n$ there is a Comtet function of the form $\frac{1}{2} + 1/(24m) - (1/(48m^2))(1 + o(1))$. For $A \geq 2$ there is a Comtet function between $\frac{1}{2} + 1/(24m) - 1/(49m^2)$ and $\frac{1}{2} + 1/(24m) - 1/(47m^2)$.

For larger values of $A$ the coefficients of $m^{-2}$ can be taken much closer together.

Theorem 3 does not disprove the conjecture that $\omega(A) = \frac{1}{2}$ is a Comtet function for the harmonic series, but it does seem to make it less plausible. It is conceivable that the fractional part of $e^A - \gamma$ never falls between $\frac{1}{2}$ and $\frac{1}{2} + 1/(24m) - 1/(48m^2)$). A machine computation for $A = 20(1)200$ found no exceptions; in fact, the cruder Comtet function found in [2] was more than adequate to determine $n_A$ for $A \leq 200$. The values of $n_A$ for $A = 1(1)20$ are given in [2] and reproduced in [9], sequence 1385; $n_{21}$ and $n_{22}$, calculated by H. P. Robinson, are given in a supplement to [9]. After the present paper had been submitted for publication, Robert Spira communicated to me the results of his computations in which he obtained $n_A$ for $A = 100(100)1000$, and also showed that there are no exceptions to the conjecture for $A \leq 1000$. Since $1/(24m)$ is about $2 \times 10^{-436}$ at this point, any exception to the conjecture will have the fractional part of $e^A - \gamma$ closer to $\frac{1}{2}$ than this, so that it seems unlikely that the conjecture will be disproved by computation.

For the series $\Sigma n^{-\gamma}$, the corresponding conjecture is that $n_A$ is the closest integer to $(A - \gamma + 2)^2/4$, where now $\gamma = 0.53964 54911 9 = 2 + \varepsilon(3/2)$ (as pointed out to me by John W. Wrench, Jr., who also provided me with the decimal approximation). I found no exceptions for $A = 2(1)1000$. 

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\[
\begin{align*}
(1) & \sum_{1}^{\infty} \frac{1}{\log \log(n + 2)} \\
(2) & \sum_{1}^{\infty} \frac{1}{\log(n + 1)} \\
(3) & \sum_{1}^{\infty} \frac{1}{n^6} \\
(4) & \sum_{1}^{\infty} \frac{1}{n} \\
(5) & \sum_{1}^{\infty} \frac{1}{(n + 1)\log(n + 1)} \\
(6) & \sum_{1}^{\infty} \frac{1}{(n + 2)\log(n + 2)\log\log(n + 2)}
\end{align*}
\]

<table>
<thead>
<tr>
<th>Series</th>
<th>( \gamma )</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>10</th>
<th>20</th>
<th>100</th>
<th>1000</th>
<th>100000</th>
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<tr>
<td>1</td>
<td>7.21848</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>6</td>
<td>112</td>
<td>1812</td>
<td>2.62 \times 10^6 (a)</td>
</tr>
<tr>
<td>2(b)</td>
<td>0.80193</td>
<td>3</td>
<td>5</td>
<td>7</td>
<td>9</td>
<td>12</td>
<td>20</td>
<td>56</td>
<td>489</td>
<td>7764</td>
<td>1.55 \times 10^7</td>
</tr>
<tr>
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<td>0.53964549</td>
<td>5</td>
<td>7</td>
<td>10</td>
<td>14</td>
<td>18</td>
<td>33</td>
<td>115</td>
<td>2574</td>
<td>250731</td>
<td>2.50 \times 10^{11} (c)</td>
</tr>
<tr>
<td>4</td>
<td>0.57721566</td>
<td>11</td>
<td>31</td>
<td>83</td>
<td>227</td>
<td>616</td>
<td>12367</td>
<td>2.7 \times 10^8</td>
<td>1.5 \times 10^{4.3}</td>
<td>1.1 \times 10^{4.34}</td>
<td>( T(4.3 \times 10^5) )</td>
</tr>
<tr>
<td>5</td>
<td>0.42816572</td>
<td>8717</td>
<td>5.1 \times 10^{10}</td>
<td>1.3 \times 10^{29}</td>
<td>1.4 \times 10^{29}</td>
<td>1.4 \times 10^{215}</td>
<td>1.6 \times 10^{4321}</td>
<td>T_2(8)</td>
<td>T(5 \times 10^{4.2})</td>
<td>T(4 \times 10^{4.33})</td>
<td>T_2(4.3 \times 10^5)</td>
</tr>
<tr>
<td>6</td>
<td>2.29992697</td>
<td>1</td>
<td>3</td>
<td>56</td>
<td>3.1 \times 10^{19}</td>
<td>T(1.3 \times 10^{4})</td>
<td>T(7 \times 10^{8.9})</td>
<td>T_2(2 \times 10^6)</td>
<td>T_2(1.1 \times 10^{41})</td>
<td>T_2(8 \times 10^{431})</td>
<td>T_2(4.3 \times 10^5)</td>
</tr>
</tbody>
</table>

Notes: To simplify the typography, I write \( T(x) = T_1(x) = 10^x \), \( T_n(x) = T(T_{n-1}(x)) \).

(a) The function \( \varphi \) for series 1 has apparently not been tabulated before; I tabulated it in order to get \( \gamma \) and \( \psi(A - \gamma) \). The value \( 2.6 \times 10^6 \) given in [4] corresponding to \( A = 10^6 \) was probably arrived at by arguing that \( \varphi(x) \) is nearly \( x/\log\log x \), so \( \psi(x) \) is nearly \( x \log \log x \).

(b) Here \( \varphi(x) \) was sufficiently well tabulated [5], [7], [8].

(c) It is easy to find this entry exactly.
I am indebted to Dr. Wrench for the 150D value of $e^{-7}$ which made the computations for the harmonic series possible. I am also indebted to Lester M. Carlyle, Jr., for communicating the results of his calculations which suggested the possibility of a result like Theorem 3.

I take this opportunity to note the following errata to [2]: In Theorem 1, last line, read $m$ for $n$ (twice). On p. 866, in the line before formula (1), read $-\frac{1}{8} n^{-2}$. On p. 868, lines 9 and 10 (statements (ii) and (iii)) read $m$ for $n$. On p. 865, first line, read “for $A = 5, 10, 100$ his values are somewhat inaccurate.”

2. Proof of Theorems 1 and 2. By the Euler-Maclaurin formula we can write

$$s_n = \gamma + \varphi(n) + \frac{1}{2} f(n) + \frac{1}{12} f'(n) + R_n,$$

where

$$R_n = -\int_n^\infty f'''(t)P_3(t) \, dt,$$

and $P_3$ is the function of period 1 that is equal on $(0, 1)$ to the Bernoulli polynomial $B_3(x)/6$. (Notation for the $B$'s as in [6] or [1].) We can estimate $R_n$ as in [6, pp. 538-539]; it turns out that

$$0 < R_n < \frac{1}{720} |f'''(n)| = O(f(n)/n^3).$$

Suppose now that $n$ is any integer such that $s_n > A$. Put $\delta_n = \frac{1}{2} f(n) + f'(n)/12 + R_n$; then from (2.1) we have $\varphi(n) + \delta_n > A - \gamma$, whence

$$\psi(\varphi(n) + \delta_n) > \psi(A - \gamma).$$

We have $\varphi(n) \to \infty$ and $\delta_n \to 0$, so that it is reasonable to expand the left-hand side of (2.3) in a Taylor series with remainder of order 3,

$$\psi(\varphi(n) + \delta_n) = \psi(\varphi(n)) + \delta_n \psi'(\varphi(n)) + \frac{1}{2} \delta_n^2 \psi''(\varphi(n)) + E_n,$$

where we may assume that

$$|E_n| \leq \frac{1}{6} \delta_n^3 \max \{ |\psi'''(\varphi(n))|, |\psi''(\varphi(n + 1))| \},$$

when $n$ is large enough (since we assumed that $|\psi'''|$ is monotonic). But $\psi(\varphi(n)) = n$, $\psi'(\varphi(n)) = 1/f(n)$, $\psi''(\varphi(n)) = -f'(n)/f(n)^3 = O(n^{-1} f(n)^{-2})$, and

$$\psi'''(\varphi(n)) = (3f'(n)^2 - f(n)f''(n))/f(n)^5 = O(n^{-2} f(n)^{-3})$$

(and similarly for $\psi'''(\varphi(n + 1))$). Hence, (2.4) becomes

$$\psi(\varphi(n) + \delta_n) = n + \delta_n/f(n) - \frac{1}{2} \delta_n^2 f'(n)/f(n)^3 + E_n,$$

where $E_n = O(n^{-2})$.

Now write $\delta_n = \frac{1}{2} f(n) + f'(n)/12 + R_n$ and multiply out $\delta_n^2$ in (2.6). We get

$$\psi(\varphi(n) + \delta_n) = n + \frac{1}{2} - \frac{1}{24} f'(n)/f(n) + O(n^{-2}),$$

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where the $O$-term can be calculated more precisely in any particular case. Thus, if $n$ is large enough, we can combine (2.3) and (2.7) to get

$$n + \frac{1}{2} - \frac{1}{24} f'(n)/f(n) + O(n^{-2}) > \psi(A - \gamma), \quad n > \psi(A - \gamma) - \frac{1}{2} + O(n^{-1}).$$

Consequently, if $m = [\psi(A - \gamma)]$ and $A$ is large enough, we have $n > m - \frac{1}{2}$. Since $n$ is an integer, this means that $n \geq m$. Now it was assumed that $s_n \geq A$; in particular, $n$ can be $n_A$, the smallest such index, and we conclude that $n_A = m$.

Similarly, if $n = n_A - 1$, we have $s_n < A$, and so

$$n_A - 1 < \psi(A - \gamma) - \frac{1}{2} + O(n^{-1}), \quad n_A < m + \frac{3}{2} + O(n^{-1}),$$

whence $n_A \leq m + 1$.

Consequently, we have shown that $m = [\psi(A - \gamma)] \leq n_A \leq m + 1$ for large $A$, and this is the conclusion of Theorem 1.

To go further, suppose that

$$\psi(A - \gamma) > m + \frac{1}{2} + \left( \frac{1}{24} + \epsilon \right) |f'(m)|/f(m), \quad \epsilon > 0.$$

By definition, $s_n \geq A$ for $n = n_A$ and hence by (2.7), (2.3) and (2.8)

$$n_A + \frac{1}{2} + \frac{1}{24} \left| f'(n_A) \right|/f(n_A) + O(n_A^{-2}) > m + \frac{1}{2} + \left( \frac{1}{24} + \epsilon \right) |f'(m)|/f(m).$$

Thus,

$$n_A > m + \frac{1}{2} + \frac{1}{24} \left\{ |f'(m)|/f(m) - |f'(n_A)|/f(n_A) \right\} + \epsilon |f'(m)|/f(m) + O(n_A^{-2}).$$

We know that $m + 1 \geq n_A \geq m$; since $|f'(x)|/f(x)$ decreases, the expression in braces is nonnegative and so $n_A > m$ if $A$ is large enough, and (2.8) holds.

Similarly, if $s_n < A$ (as it is when $n = n_A - 1$), we have

$$n + \frac{1}{2} + \frac{1}{24} |f'(m)|/f(n) + O(n^{-2}) < \psi(A - \gamma).$$

Supposing that

$$\psi(A - \gamma) < m + \frac{1}{2} + \left( \frac{1}{24} - \epsilon \right) |f'(m)|/f(m), \quad \epsilon > 0,$$

we get

$$n < m + \frac{1}{24} \left\{ |f'(m)|/f(m) - |f'(n)|/f(n) \right\} - \epsilon |f'(m)|/f(m) + O(m^{-2}).$$

Here $n = n_A - 1 < m$, so the expression in braces is not positive and consequently $n < m$, i.e., $n_A < m + 1$. Therefore, $n_A = m$ under (2.10) if $A$ is large enough.

3. Proof of Theorem 3. We have $\varphi(x) = \log x$, $\psi(x) = e^x$, $\gamma = 0.57721 56649 \ldots$. Then (2.1) becomes

$$s_n = \gamma + \log n + \frac{1}{2n} - \frac{1}{12n^2} + R_n,$$

where
and by (2.2)

\[ 0 < R_n < \frac{1}{120} n^{-4}. \]

We now proceed as in Theorem 1 but take one more term in the Taylor series for \( \psi(x) = e^x \). Here \( \delta_n = (2n)^{-1} - (12n^2)^{-2} + R_n \) and

\[
\psi(\varphi(n) + \delta_n) = ne^{\delta_n} = n \left( 1 + \delta_n + \frac{1}{2} \delta_n^2 + \frac{1}{6} \delta_n^3 + \epsilon_n n^{-4} \right),
\]

where

\[ 0 < \epsilon_n \leq \frac{1}{24} e^{\delta_n} \delta_n^4 < \frac{1}{384} e^{1/(2n)} < 0.0034 \]

if \( n \geq 2 \). Expanding the powers of \( \delta_n \), we get

\[
\psi(\varphi(n) + \delta_n) = n \left\{ 1 + \frac{1}{2} n^{-1} - \frac{1}{12} n^{-2} + R_n + \frac{1}{12} n^{-3} + \frac{1}{720} n^{-4} + R_n \right. \\
\left. + R_n \left( \frac{1}{8} n^{-3} + \frac{1}{4} n^{-2} + \frac{1}{12} n^{-1} + \frac{1}{24} n^0 + \frac{1}{120} n^{-2} \right) \right. \\
+ \left. \left( \frac{1}{2} n^{-1} - \frac{1}{24} n^{-2} + \frac{1}{120} n^{-3} \right) \right. \\
+ \left. \left( \frac{1}{2} n^{-1} - \frac{1}{24} n^{-2} + \frac{1}{120} n^{-3} \right) \right. \\
= n + \frac{1}{2} n^{-1} - \frac{1}{24} n^{-2} + E_n,
\]

where

\[
n^3 E_n = \frac{1}{6} \epsilon_n \left[ 1 + \frac{1}{2} n^{-1} - \frac{1}{12} n^{-2} + \frac{1}{144} n^{-4} + \frac{1}{576} n^{-6} + \frac{1}{10368} n^{-8} + \frac{1}{41472} n^{-10} \right]
\]

Since each of the expressions in parentheses is positive for \( n \geq 2 \), we get an upper bound for \( n^3 E_n \) by replacing \( \epsilon_n \) and \( R_n \) by their upper bounds from (3.1) and (3.2). The result is a decreasing function of \( n \), so it is largest at \( n = 2 \) and we get, after some calculation, \( n^3 E_n < 0.005 \). To get a lower bound for \( n^3 E_n \) we have only to replace \( R_n \) and \( \epsilon_n \) by 0, and then we get

\[
n^3 E_n > - \frac{1}{144} - \frac{1}{41472} > -0.007.
\]

Using the upper bound, we obtain, for \( n = n_A \),

\[
n + \frac{1}{2} + \frac{1}{24} n^{-1} - \frac{1}{48} n^{-2} + 0.005 n^{-3} > e^{A-\gamma}.
\]

Consequently, with \( m = [e^{A-\gamma}] \), if
we have

\begin{equation}
(3.4) \quad n > m + \frac{1}{24} (m^{-1} - n^{-1}) + \frac{1}{48} (n^{-2} - m^{-2}) + \left(\frac{1}{48} - \frac{1}{49}\right)m^{-2} - 0.005n^{-3}.
\end{equation}

But we know that \(n \geq m\); if we had \(n = m\), (3.4) would yield

\[ 0 > \left(\frac{1}{48} - \frac{1}{49}\right)m^{-2} - 0.005m^{-3}. \]

Now suppose that \(A > 4\); then \(m = [e^{A-\gamma}] \geq 30\), and so we would have

\[ 0 > \left(\frac{1}{48} - \frac{1}{49}\right) - (0.005)/30 > 0.000425 - 0.00016. \]

This contradiction shows that \(n > m\), so that \(n = n_A = m + 1\) under (3.3).

On the other hand, with \(n = n_A - 1\) we have

\[ n + \frac{1}{2} + \frac{1}{24} n^{-1} - \frac{1}{48} n^{-2} - 0.007n^{-3} < e^{A-\gamma}. \]

If \(n < m\), we have \(n_A < m + 1\) and so \(n_A = m\), so we have only to exclude the possibility that \(n = m\). If we suppose that \(n = m\) and

\begin{equation}
(3.5) \quad e^{A-\gamma} < m + \frac{1}{2} + \frac{1}{24} m^{-1} - \frac{1}{47} m^{-2},
\end{equation}

we then have

\[ m + \frac{1}{2} + \frac{1}{24} m^{-1} - \frac{1}{48} m^{-2} - 0.007m^{-3} < m + \frac{1}{2} + \frac{1}{24} m^{-1} - \frac{1}{47} m^{-2}, \]

that is, \(1/47 - 1/48 < 0.007m^{-1}\). If \(A > 4\), we again have \(m \geq 30\), and the last inequality says that \(0.00043 < 0.00024\). Thus, the assumption that \(n_A = m + 1\) leads to a contradiction if (3.5) holds.

This establishes the second part of the theorem for \(A > 4\); but it also holds, by direct computation, for \(A = 2, 3\).

If we replace \(1/47\) and \(1/49\) by \(1/48 \pm \epsilon\), we can take \(\epsilon\) as small as we like if we take \(A > A_0\), sufficiently large, and the first part of Theorem 3 follows.

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