Numbers Generated by the Reciprocal of $e^x - x - 1$

By F. T. Howard

Abstract. In this paper we examine the polynomials $A_n(z)$ and the rational numbers $A_n = A_n(0)$ defined by means of

$$e^{x} x^2 e^{x} - x - 1)^{-1} = 2 \sum_{n=0}^{\infty} A_n(z) x^n / n!.$$  

We prove that the numbers $A_n$ are related to the Stirling numbers and associated Stirling numbers of the second kind, and we show that this relationship appears to be a logical extension of a similar relationship involving Bernoulli and Stirling numbers. Other similarities between $A_n$ and the Bernoulli numbers are pointed out. We also reexamine and extend previous results concerning $A_n$ and $A_n(z)$. In particular, it has been conjectured that $A_n$ has the same sign as $-\cos n\theta$, where $\theta$ is the zero of $e^x - x - 1$ with smallest absolute value. We verify this for $1 < n < 14329$ and show that if the conjecture is not true for $A_n$, then $|\cos n\theta| < 10^{-(n-1)/5}$. We also show that $A_n(z)$ has no integer roots, and in the interval $[0, 1)$, $A_n(z)$ has either two or three real roots.

1. Introduction. Define the rational numbers $A_0, A_1, A_2, \ldots$ by means of

$$(\sum_{n=0}^{\infty} \frac{2x^n}{(n + 2)!})^{-1} = \frac{x^2/2}{e^x - x - 1} = \sum_{n=0}^{\infty} A_n x^n / n!.$$  

This definition is apparently due to L. Carlitz [4], who raised the question of whether a theorem like the Staudt-Clausen theorem holds for the numbers $A_n$. Because of the obvious similarity of (1.1) to the definition of the Bernoulli numbers $B_n$, i.e.

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n x^n / n!,$$

this seems to be a reasonable question. The writer [6] has shown, however, that evidently such a theorem does not hold: If $p$ is any odd prime, then

$$p^m A_{m(p-2)} / [m(p-2)]! \equiv 2^m \pmod{p},$$

which implies that arbitrarily large powers of $p$ will divide the denominator of some $A_n$. However, for $n > 1$,

$$2A_n \equiv 1 \pmod{4},$$

so the denominator of $A_n$, for $n > 1$, is even and not divisible by 4. This last property is also true of the Bernoulli numbers $B_{2n}$.

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In the present paper we reexamine questions raised in [6] and [8] about \( A_n \), and we attempt to clarify and extend the results in those papers. We also prove that the numbers \( A_n \) are related to the Stirling numbers of the second kind, and we show that this relationship appears to be a logical extension of a similar relationship involving Bernoulli and Stirling numbers. The goal of the present paper is to show that (1.1) is a natural definition to make and that the \( A_n \) are of interest in their own right. A summary by sections follows.

In Section 2 we examine a conjecture made in [8] about the sign of \( A_n \). We prove that if \( re^{i\theta} \) is the zero of \( e^x - x - 1 \) with smallest absolute value, then \( A_n \) has the same sign as \(-\cos n\theta\) if \(|\cos n\theta| > 10^{-(n-1)/5}\). We show that \( A_n \) does indeed have the same sign as \(-\cos n\theta\) for \( 1 \leq n \leq 14329 \).

In Section 3 we examine the polynomials \( A_n(z) \), defined in [6] by means of

\[
\frac{(x^2/2)e^{xz}}{e^x - x - 1} = \sum_{n=0}^{\infty} A_n(z) \frac{x^n}{n!}.
\]

We prove that if \( n > 1 \), \( A_n(z) \) has either two or three real roots in the closed interval \([0, 1]\). We show that \( A_n(z) \) has no integer roots and \( A_{2n}(z) \) has no rational roots. For special values of \( n \) we show \( A_n(z) \) is irreducible over the rational field.

In Section 4 we prove some general theorems for numbers generated by the reciprocal of any series. We show that, in a sense, there is always an explicit formula for these numbers, and there is also a way of expressing these numbers as a linear combination of numbers which have a combinatorial interpretation.

In Section 5 we apply the theorems of Section 4 to \( A_n \). We show how \( A_n \) can be expressed in terms of the Stirling numbers of the second kind and the associated Stirling numbers of the second kind.

In Section 6 we prove some miscellaneous results for \( A_n \) and \( A_n(z) \). We show that \( 2|A_n| < |B_n| \), if \( n \) is even; and we prove some theorems concerning possible rational roots of \( A_n(z) \), if \( n \) is odd. We include in this section a table of values of \( \cos n\theta \), \( 1 \leq n \leq 46 \), where \( re^{i\theta} \) is the zero of \( e^x - x - 1 \) with smallest absolute value. We also include a table indicating how the sign of \( A_n \) changes for \( 1 \leq n \leq 14329 \).

All calculations in this paper were performed on a Texas Instruments SR-50A calculator. This machine computes to thirteen significant digits and rounds off to ten significant digits.

We note that a listing of the first 15 numbers \( A_n \) can be found in [6].

2. Sign of \( A_n \). It is pointed out in [8] that by using Hadamard's factorization theorem [17, p. 250], we can write \( 2A_n = -n! \Sigma_{s=1}^{\infty} (x_s)^{-n} \), where \( x_1, x_2, \ldots \) are the zeros of \( e^x - x - 1 \). Using

\[
x_s = r_s e^{i\theta_s}, \quad \overline{x_s} = r_s e^{-i\theta_s},
\]

we can write
(2.1) \[ A_n = -n! \sum_{s=1}^{\infty} r_s^{-n} \cos n\theta_s. \]

If we let \( x_1 \) be the zero with smallest absolute value, the following conjecture was made in [8].

**Conjecture.** For \( n > 0 \), \( A_n \) has the same sign as \( -\cos n\theta_1 \).

We shall refer to this as "the sign conjecture", and we shall show that it is true at least for \( n \leq 14329 \). In [8] the conjecture was verified for \( n \leq 37 \).

It is not too difficult to find approximations for \( x_s \). If we set \( e^x - x - 1 = 0 \) and let \( x = a + bi \), we see that

\[ a = b \cot b - 1 = \ln b - \ln(\sin b), \quad (\sin b) \exp(b \cot b) = eb; \]

and by examining the graphs of \( e^x \) and \( (\sin x) \exp(x \cot x) \), we see that

\[ (2n + 1/4)\pi < b < (2n + 1/2)\pi, \quad n = 1, 2, \ldots . \]

We can compute the following approximations: \( x_1 = a + bi \), with

\[ 2.08884300 < a < 2.08884302, \]
\[ 7.461489270 < b < 7.461489300, \]
\[ 74.360416° < \theta_1 < 74.360417°, \]
\[ 7.748360 < r_1 < 7.748361. \]

From (2.1) we see that \( A_n \) has the same sign as \( -\cos n\theta_1 \) if

\[ |\cos n\theta_1| > \left| \sum_{s=2}^{\infty} \left( \frac{r_1}{r_s} \right)^n \cos n\theta_s \right|. \]

Since, by (2.2) and (2.3),

\[ \sum_{s=2}^{\infty} \left( \frac{r_1}{r_s} \right)^n < \sum_{s=2}^{\infty} \left( \frac{7.75}{2\pi s} \right)^n < \left( \frac{5}{4} \right)^n \sum_{s=2}^{\infty} s^{-n}, \]

we have the following theorem.

**Theorem 2.1.** If \( |\cos n\theta_1| > (5/4)^n \sum_{s=2}^{\infty} s^{-n} \), then \( A_n \) has the same sign as \( -\cos n\theta_1 \).

The sum in Theorem 2.1 is very small for large \( n \). In fact, it is not difficult to show it is less than \((5/8)^n - 1\) and hence less than \(10^{-(n-1)/5}\).

**Corollary.** If \( |\cos n\theta_1| > 10^{-(n-1)/5} \), then \( A_n \) has the same sign as \( -\cos n\theta_1 \).

The values of \( \cos n\theta_1 \) have been computed for \( 1 \leq n \leq 46 \) and are included in Section 6. We see that the sign conjecture holds for \( 1 \leq n \leq 46 \), the smallest value of \( \cos n\theta_1 \), being .005 when \( n = 23 \). We have the following approximations modulo 360 degrees.

\[ 23\theta_1 = 270.289°, \quad 46\theta_1 = 180.579°, \]
\[ 69\theta_1 = 90.869°, \quad 92\theta_1 = 1.158°. \]

We see that if \( n = 46 + k, 0 < k < 46 \), then \( A_n \) and \( A_k \) have different signs; the exact opposite of the original pattern of signs occurs for \( 46 < n < 92 \). \( A_0 \) is a special case
for which the sign conjecture is not true.) In fact, we expect \( A_{46 + k} / (46 + k)! \) to be approximately \(-r_1^{-46} A_k / k!\). Also, the pattern of signs for \(0 < n \leq 92\) will be repeated for \(92 < n \leq 184\); that is, \( A_{92 + k} \) and \( A_k \) will have the same sign for \(k = 1, 2, \ldots, 92\). The following theorem tells exactly what the signs are for \(1 < n \leq 327\).

**Theorem 2.2.** For positive \(n\), let \(n = 46k + s\), \(0 < k < 6\), \(0 < s < 46\). Let 
\[
s = 12m + t, \quad 0 \leq t < 12.
\]
If \(t = 0, 1, 4, 5, 6, 9 \text{ or } 10\), then \((-1)^{k+m+1} A_n > 0\). If \(t = 2, 3, 7 \text{ or } 8\), then \((-1)^{k+m} A_n > 0\). If \(s = 11 \text{ or } 23\), then \((-1)^{k+m} A_n > 0\). If \(s = 35\), then \((-1)^{k+1} A_n > 0\).

As \(n\) gets larger, the discrepancy between \(46\theta_1\) and 180 degrees begins to make a difference. Using Table 1 in Section 6, we see that the first change in the pattern of Theorem 2.2 occurs at \(n = 328 = 46 \cdot 7 + 6\). That is, as \(k\) increases from 0 to 7, the angle \((46k + 6)\theta_1\) changes in the following way (approximately): \(86^\circ, 267^\circ, 87^\circ, 268^\circ, \ldots\). If \(n = 46k + s\), \(7 < k < 12\), the pattern of Theorem 2.2 holds with one exception: if \(n = 46k + 6\), then \((-1)^k A_n > 0\). As \(n\) gets larger, the pattern will continue to change. Table 2 in Section 6 indicates when the pattern of Theorem 2.2 changes for various values of \(s\). When \(n = 633 = 46 \cdot 13 + 35\), for example, the pattern changes for numbers of the form \(46k + 35\); i.e. \((-1)^k A_{46k+35} > 0\).

By checking the value of \(\cos n\theta_1\) at the numbers given in Table 2, and also at \(n = 46(k - 1) + s\), we see, by the corollary to Theorem 2.1, that \(A_n\) has the same sign as \(-\cos n\theta_1\) for \(1 \leq n \leq 14329\). The smallest value of \(\cos n\theta_1\) for \(1 \leq n \leq 14329\) occurs when \(n = 1243\) and is about \(.00004\). We have used the approximation \(74.360416 < \theta_1 < 74.360417\) in these calculations. We see by the corollary to Theorem 2.1 that if the sign conjecture is not true for \(A_n\), then \(|\cos n\theta_1| < 10^{-2865}\).

**Theorem 2.3.** For \(n > 0\), we never have \(A_n > 0, A_{n+1} < 0, A_{n+2} > 0\) or \(A_n < 0, A_{n+1} > 0, A_{n+2} < 0\).

**Proof.** Suppose \(A_n > 0, A_{n+1} < 0, A_{n+2} > 0\). Since \(\theta_1\) is about 74 degrees, it is clear the sign conjecture does not hold for at least one of \(n, n+1\) or \(n+2\).

Suppose \(A_n\) does not have the same sign as \(-\cos n\theta_1\). Then by the corollary to Theorem 2.1, \(n\theta_1\) is within one degree (modulo 360 degrees) of either 90 or 270 degrees. It is then clear that the sign conjecture does hold for \(A_{n+1}\) and \(A_{n+2}\), and in fact, they both must have the same sign, which is a contradiction. If the sign conjecture does not holds for \(A_{n+1}\), we see that \(A_n\) and \(A_{n+2}\) must have opposite signs, and if the sign conjecture is not true for \(A_{n+2}\), we see that \(A_n\) and \(A_{n+1}\) must have the same sign. The reasoning is similar if \(A_n < 0, A_{n+1} > 0, A_{n+2} < 0\).

Using the same kind of reasoning, we have the following theorem.

**Theorem 2.4.** For \(n \geq 0\), we never have four consecutive numbers \(A_n, A_{n+1}, A_{n+2}, A_{n+3}\) with the same sign.

Because of (2.1) and the fact that
\[
\sum_{s=2}^{\infty} (r_1/r_s)^n < (5/8)^{n-1},
\]
we see that, for \(n \geq 20\), if \(|\cos (n + 1)\theta_1| - r_1 |\cos n\theta_1| > .001\), then \(|A_{n+1}| > (n + 1)|A_n|\). On the other hand, if \(r_1 |\cos n\theta_1| > 1.001\), then \((n + 1)|A_n| > |A_{n+1}|\). Thus we have the following theorem, which actually holds for all \(n \geq 0\).
Theorem 2.5. If $|\cos n\theta_1| < .118$, then $|A_{n+1}| > (n+1)|A_n|$. If $|\cos n\theta_1| > .1292$, then $(n+1)|A_n| > |A_{n+1}|$.

Usually $(n+1)|A_n| > |A_{n+1}|$, but this is not true for many values of $n$ including

\begin{align*}
n &= 46k + 6, \quad 0 \leq k \leq 6, \\
n &= 46k + 35, \quad 2 \leq k \leq 12, \\
n &= 46k + 18, \quad 6 \leq k \leq 19.
\end{align*}

For these particular values of $n$, $A_n$ and $A_{n+1}$ have opposite signs, a fact that is important when we are examining the real roots of $A_{n+1}(z)$. Of course there are cases, like $n = 23$, when $A_n$ and $A_{n+1}$ have the same sign and $(n+1)|A_n| < |A_{n+1}|$.

3. The Polynomials $A_n(z)$. It was proved in [8] that the polynomial $A_n(z)$ defined by (1.5) has at least one real root in the closed interval $[0, 1]$ for $n > 0$. In this section we show that $A_n(z)$ has either two or three real roots in $[0, 1]$, and in addition we prove that $A_{2n}(z)$ has no rational roots for $n > 0$. For a few specific values of $n$, we show that $A_n(z)$ is irreducible over the rational field. These results can be compared to similar properties of the Bernoulli and Euler polynomials [1], [2], [9], [10], [15].

In [6] the following formulas were proved.

\begin{align*}
(3.1) & \quad A_n(z) = \sum_{r=0}^{n} \binom{n}{r} A_r z^{n-r}, \\
(3.2) & \quad A'_n(z) = nA_{n-1}(z), \\
(3.3) & \quad A_n(z + 1) - A_n(z) - A'_n(z) = \binom{n}{2} z^{n-2} \quad \text{for} \quad n > 1.
\end{align*}

It follows from (3.2) and (3.3) that

\begin{equation}
(3.4) \quad \int_0^1 A_n(z) \, dz = A_n,
\end{equation}

and more generally

\begin{equation}
(3.5) \quad \int_y^{y+1} A_n(z) \, dz = A_n(y) + ny^{n-1}/2.
\end{equation}

In the theorems that follow, we assume $u/b$ is a rational number reduced to its lowest terms. Also, we note that

\begin{align*}
A_0(z) &= 1, \quad A_1(z) = z - 1/3,
\end{align*}

so $A_1(z)$ does have the rational root $1/3$.

Theorem 3.1. If $A_n(u/b) = 0$, then $b = 3$ and $u \equiv n \equiv 1 \pmod{3}$.

Proof. By (3.1) we have

\begin{equation}
\frac{3^n}{n!} A_n(z) = \sum_{r=0}^{n} 3^r \frac{3^r}{r!} A_r \frac{3^{n-r}}{(n-r)!} z^{n-r},
\end{equation}

and since $3^{n-r}/(n-r)! \equiv 0 \pmod{3}$, unless $r = n$, we have, by (1.3),

\begin{equation}
3^n A_n(z)/n! \equiv (-1)^n \pmod{3}.
\end{equation}

It follows that if $u/b$ is a root then $b \equiv 0 \pmod{3}$. Otherwise we have $(-1)^n \equiv 0$.
(mod 3). (See also Lemma 2.3 in [7].) We have, from (3.1),

\[(3.6) \quad 0 = \frac{u^n}{b} - \frac{nu^{n-1}}{3} + \binom{n}{2} \frac{u^{n-2}b}{18} + \sum_{r=3}^{n} \binom{n}{r} A_r u^{n-r} b^{r-1}.\]

In [6] it is shown that if \( m = \lceil n/(p - 2) \rceil + 1 \), \( p \) an odd prime, then \( p^m A_n/n! \equiv 0 \pmod{p} \). Thus \( b^{r-1} A_r \) is integral (mod \( b \)) for \( r > 2 \), and we see that

\[(3.7) \quad \frac{u^n}{b} - \frac{nu^{n-1}}{3} + \binom{n}{2} \frac{u^{n-2}b}{18} \]

must be integral (mod 3); i.e., the above sum is a rational number with denomination not divisible by 3. For any prime \( p \neq 3 \), let \( p^s \) be the highest power of \( p \) dividing \( b \). Then if \( s > 0 \),

\[0 \equiv p^s u^n/b \not\equiv 0 \pmod{p},\]

by (3.6), which is impossible. Now suppose \( b = 3^s \). If \( s > 1 \), we see from (3.6) that \( 0 \equiv u^n \pmod{3} \), a contradiction since g.c.d. \((u, 3) = 1\). Hence \( b = 3 \), and since (3.7) must be integral (mod 3), we must have \( u \equiv n \equiv 1 \pmod{3} \).

Theorem 3.1 shows that no polynomial \( A_n(z) \) has an integer root.

**Theorem 3.2.** For \( n \geq 0 \), \( A_{2n}(z) \) has no rational roots.

**Proof.** By (1.4) and (3.1), we have, for any \( k \geq 2 \),

\[2A_k(z) \equiv \sum_{r=2}^{k} \binom{k}{r} z^{k-r} + 2z^k + 2kz^{k-1} \equiv (1 + z)^k + z^k + kz^{k-1} \pmod{4}.\]

If \( k = 2n \), we see that \( 2A_{2n}(u/3) \equiv 1 \pmod{2} \), so \( u/3 \) cannot be a root of \( A_{2n}(z) \).

Unfortunately, it is not clear whether or not \( A_{2n+1}(z) \) can have rational roots.

If we let \( k = 2n + 1 \) in the proof of Theorem 3.2, the only conclusion we can draw is that \( u \) is odd and \( u \equiv 2n + 1 \pmod{4} \). We do know by Theorems 3.1 and 3.2 that if \( A_n(u/3) = 0 \), then \( n \equiv 1 \pmod{6} \). Furthermore, it can be proved that if \( p - 2 \) divides \( n \), where \( p \) is any prime number larger than 3, then \( A_n(z) \) does not have a rational root. Also, if \( A_n(1/3) = 0 \), \( n > 1 \), then \( n \equiv 1 \pmod{36} \). These last two results are proved in Section 6.

Next we examine the real roots of \( A_n(z) \) on the closed interval \([0, 1]\).

**Lemma 3.1.** If \( n > 1 \), then \( A_n(z) \) has at least two real roots in \([0, 1]\).

**Proof.** We shall consider four different cases, using (3.2), (3.3), (3.4).

**Case 1.** \( A_n > 0, A_{n+1} > 0 \). We see that \( A_{n+1}(z) \) is an increasing function at \( z = 0 \) and that \( A_{n+1}(1) > A_{n+1}(0) \). It follows from (3.4) that the area bounded by \( A_{n+1}(z) \), the \( x \)-axis and the lines \( x = 0, x = 1 \) is exactly \( A_{n+1} = A_{n+1}(0) \). Thus for some values of \( z \) we must have \( A_{n+1}(z) < A_{n+1} \), and we see there must be at least two "critical points" on the graph, i.e., there are two real numbers \( a \) and \( b \), \( 0 < a < b < 1 \), such that \( 0 = A'_{n+1}(a) = A'_{n+1}(b) \). Thus \( A_n(a) = 0 = A_n(b) \). The case \( A_n < 0, A_{n+1} < 0 \) is similar.

**Case 2.** \( A_n < 0, A_{n+1} > 0 \). In this case \( A_{n+1}(1) < A_{n+1}(0) \) and \( A_{n+1}(z) \) is a
Lemma 3.2. If \( n \geq 0 \), then \( A_n(z) \) has no more than three real roots in \( [0, 1] \).

Proof. Suppose \( n \) is the smallest positive integer such that \( A_n(z) \) has more than three real roots in \( [0, 1] \). Then \( n > 3 \).

Case 1. \( A_n > 0, A_{n-1} > 0 \). Since \( A_n(z) \) is increasing at \( z = 0 \), we see that there must be at least four critical points on the graph of \( A_n(z) \). This implies that \( A_{n-1}(z) \) has at least four real roots in \( [0, 1] \), a contradiction. The case \( A_n < 0, A_{n-1} < 0 \) is similar. It is clear that if the lemma is true for \( A_n(z) \), and \( A_n \) and \( A_{n-1} \) have the same sign, then \( A_n(z) \) has exactly two real roots in \( [0, 1] \).

Case 2. \( A_n > 0, A_{n-1} < 0, A_n(1) < 0 \). If \( A_n(z) \) has at least four real roots in \( [0, 1] \), it is clear there are at least four critical points on the graph of \( A_n(z) \). This implies \( A_{n-1}(z) \) has at least four real roots in \( [0, 1] \), a contradiction. The case \( A_n < 0, A_{n-1} > 0, A_n(1) > 0 \) is similar.

Case 3. \( A_n > 0, A_{n-1} < 0, A_n(1) > 0 \). By Theorem 2.3 we know \( A_{n-2} < 0 \), and from Case 1 we know \( A_{n-1}(z) \) has exactly two real roots in \( [0, 1] \). If \( A_n(z) \) has at least four real roots in \( [0, 1] \), there are at least three critical points on the graph of \( A_n(z) \), which is impossible. The case \( A_n < 0, A_{n-1} > 0, A_n(1) < 0 \) is similar.

Lemma 3.3. If \( n \geq 0 \), \( A_n(z) \) has no multiple real roots in \( [0, 1] \).

Proof. Suppose \( n \) is the smallest positive integer such that \( A_n(z) \) has a multiple root. By (3.2) it must be a double root.

Case 1. \( A_n > 0, A_{n-1} > 0 \). We know \( A_n(z) \) is increasing at \( z = 0 \); \( A_n(1) > A_n(0) \), and \( A_n(z) \) has exactly two distinct real roots in \( [0, 1] \). We see, then, that a double root implies four critical points on the graph of \( A_n(z) \), a contradiction. The case \( A_n < 0, A_{n-1} < 0 \) is similar.

Case 2. \( A_n > 0, A_{n-1} < 0, A_n(1) < 0 \). The only possibility is that \( A_n(z) \) has exactly two real roots in \( [0, 1] \), one of them a double root. By Theorem 2.3, we know \( A_{n+1} > 0 \), so \( A_{n+1}(z) \) has exactly two real roots in \( [0, 1] \). Also, \( A_{n+1}(z) \) is decreasing at \( z = 1 \), since \( A_n(1) < 0 \), and is increasing at \( z = 0 \). This implies there are at least three critical points on the graph of \( A_{n+1}(z) \), a contradiction. The case \( A_n < 0, A_{n-1} > 0, A_n(1) > 0 \) is similar.

Case 3. \( A_n > 0, A_{n-1} < 0, A_n(1) > 0 \). Since \( A_n(z) \) has at least two distinct real roots in \( [0, 1] \), a double root implies at least three critical points on the graph of \( A_n(z) \). We know, however, that \( A_{n-1} \) has exactly two real roots in \( [0, 1] \) since \( A_{n-2} < 0 \). The case \( A_n < 0, A_{n-1} > 0, A_n(1) < 0 \) is similar.

By Lemmas 3.1, 3.2 and 3.3, we have the following theorem.

Theorem 3.3. Suppose \( n > 1 \). Then \( A_n(z) \) has no multiple real roots in \( [0, 1] \), and

(a) if \( A_n \) and \( A_{n-1} \) have the same sign, then \( A_n(z) \) has exactly two real roots in \( [0, 1] \).

(b) if \( A_n \) and \( A_{n-1} \) have opposite signs, and if \( n|A_{n-1}| > |A_n| \), then \( A_n(z) \) has exactly three real roots in \( [0, 1] \).
(c) if \( A_n \) and \( A_{n-1} \) have opposite signs, and if \( n|A_{n-1}| < |A_n| \), then \( A_n(z) \) has exactly two real roots in \([0, 1]\).

By (3.3), the condition \( n|A_{n-1}| > |A_n| \) is equivalent to \( A_n(1) \) having the same sign as \( A_{n-1} \), if \( A_n \) and \( A_{n-1} \) have different signs. Similarly, the condition \( n|A_{n-1}| < |A_n| \) is equivalent to \( A_n(1) \) having the same sign as \( A_n \). By Theorem 2.5 and the remarks following it, we see that usually \( A_n(1) \) has the same sign as \( A_{n-1} \). However, this is not the case for many values of \( n \), such as \( n = 46k + 6, 0 < k < 6 \).

It is not clear how the roots of \( A_n(z) \) are distributed outside the interval \([0, 1]\). If \( y > 0 \) and \( A_n(y) < 0 \), it follows from (3.5) that \( A_n(y) \) has at least one real root between \( y \) and \( y + 1 \). This is because \( A_{n+1}(z) \) is decreasing at \( z = y \) and 

\[
\int_y^{y+1} A_n(z) \, dz > A_{n+1}(y),
\]

so there must be at least one real number \( a, y < a < y + 1 \), such that \( A_{n+1}'(a) = 0 = A_n(a) \). By the same type of reasoning, if \( y < 0 \) and \( A_{2n}(z) < 0 \), then \( A_{2n}(z) \) has at least one real root between \( y - 1 \) and \( y \). If \( y < 0 \) and \( A_{2n+1}(y) > 0 \), then \( A_{2n+1}(z) \) has at least one real root between \( y - 1 \) and \( y \). The distributions of the real roots of the Bernoulli and Euler polynomials can be found in [10] and [9] respectively.

Eisenstein's irreducibility criterion has been used to show that certain Bernoulli, Euler and van der Pol polynomials are irreducible over the rational field. The same method can be used on \( A_n(z) \).

**Theorem 3.4.** If \( n = 2k, k > 0 \), or \( n = m(p - 2) \) where \( p \) is an odd prime, \( 2m < p \), then \( A_n(z) \) is irreducible over the rational field.

**Proof.** If \( n = 2k \), we have

\[
2A_n(z) = 2 \sum_{r=0}^{n} \binom{n}{r} A_r z^{n-r} \equiv 2A_n \equiv 1 \pmod{2},
\]

and furthermore \( 2A_0 \not\equiv 0 \pmod{4} \). Thus \( 2A_n(z) \) is an Eisenstein polynomial and is irreducible over the rational field. Suppose \( 2m < p \). From a theorem in [6], we know that if \( r \) is in any of the intervals \([0, p-2), [p, 2(p-2)), \ldots, [(m-1)p, mp-2)\), then \( A_r \) is integral \( \pmod{p} \), and also \( p^2 A_r \equiv 0 \pmod{p} \) for \( 0 \leq r \leq mp-2 \). We see, by (1.3), that if \( n = m(p-2) \) then \( pA_n \) is an Eisenstein polynomial.

4. The Reciprocal of a Series. In this section we prove some theorems that are true for the reciprocal of any power series. Some of our results can be proved by using generalized chain rule differentiation formulas; instead we shall generalize methods used by Jordan [12] and Riordan [16]. We do not claim these results are new, though references are somewhat hard to find. Perhaps [14] is a good general reference. The goal of this and the subsequent section is to show how the numbers \( A_n \) are related to the Stirling numbers, and associated Stirling numbers, of the second kind.

Suppose \( a_0 + a_1x + a_2x^2 + \cdots \) is a given power series, \( a_0 \not= 0 \). We shall assume that the series has a positive radius of convergence, though this condition is not really necessary for the theorems of this section. Define the numbers \( c_n \) by means of

\[
\left( \sum_{r=0}^{\infty} a_r x^r \right)^{-1} = \sum_{n=0}^{\infty} c_n x^n.
\]
NUMBERS GENERATED BY THE RECIPROCAL OF $e^x - x - 1$

Then $c_0 = 1/a_0$ and $\sum_{i=0}^{n} a_i c_{n-i} = 0$. By Cramer's rule, we have the following theorem [13, p. 116]:

**Theorem 4.1.** If $c_n$ is defined by (4.1), then

$$c_n = \frac{(-1)^n}{(a_0)^{n+1}} \begin{vmatrix} a_1 & a_0 & 0 & \cdots & 0 \\ a_2 & a_1 & a_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_n & a_{n-1} & a_{n-2} & \cdots & a_1 \end{vmatrix}.$$ 

An alternate approach is the following:

$$a_0 \sum_{n=0}^{\infty} c_n x^n = \left( \sum_{n=0}^{\infty} \frac{a_n}{a_0} x^n \right)^{-1} = \left( 1 + \sum_{n=1}^{\infty} \frac{a_n}{a_0} x^n \right)^{-1} = \sum_{j=0}^{\infty} (-1)^j \left( \sum_{n=1}^{\infty} \frac{a_n}{a_0} x^n \right)^j.$$ 

By comparing coefficients of $x$, we have the next theorem.

**Theorem 4.2.** If $c_n$ is defined by (4.1), then for $n > 0$,

$$c_n = \sum_{j=1}^{n} (-1)^j a_{k_1} \cdots a_{k_j} / (a_0)^{j+1}$$

where for each $j$ the sum is over all compositions (ordered partitions) $k_1 + \cdots + k_j = n$, each $k_i \geq 1$.

In Theorem 4.2 the order of the numbers $k_1, \ldots, k_j$ is important. For example, $1 + 3$ is not considered the same composition of 4 as $3 + 1$.

Define $F(n, j)$ by means of

$$(4.2) \sum_{r=1}^{\infty} a_r x^r = \sum_{n=j}^{\infty} j! F(n, j) \frac{x^n}{n!}.$$ 

Then

$$j! F(n, j) = n! \sum a_{k_1} \cdots a_{k_j},$$

where the sum is over all compositions $k_1 + \cdots + k_j = n$, each $k_i \geq 1$. Comparing (4.3) with Theorem 4.2, we have the next theorem.

**Theorem 4.3.** If $c_n$ is defined by (4.1) and $F(n, j)$ is defined by (4.2), then

$$n! c_n = \sum_{j=1}^{n} (-1)^j (a_0)^{-j-1} F(n, j).$$

The number $F(n, j)$ has the following interpretation [5], [16, pp. 74–78]: Consider all the partitions of the set $\{1, 2, \ldots, n\}$ into $j$ nonempty subsets (called blocks of the set partition). Assign a "weight" of $k! a_k$ to each block which has exactly $k$ elements. For each set partition there is a weight, found by multiplying the weights of the $j$ blocks making up the partition. Then $F(n, j)$ is the sum of the weights of all the set partitions of $\{1, 2, \ldots, n\}$ consisting of $j$ blocks.
For example, to compute \( F(4, 2) \), we see there are three set partitions with weight \( 4a_2 \) and four set partitions with weight \( 6a_1a_3 \). Thus \( F(4, 2) = 12a_2^2 + 24a_1a_3 \).

If we define \( F_n(s) \) by means of

\[
\sum_{n=0}^{\infty} F_n(s) \frac{x^n}{n!} = \exp \left( s \sum_{r=1}^{\infty} a_r x^r \right),
\]

we see that

\[
F_n(s) = \sum_{j=1}^{n} F(n, j)x^j.
\]

If a generating function is written in the form

\[
am^m \left( \sum_{r=m}^{\infty} a_r x^r \right)^{-1} = \sum_{n=0}^{\infty} d_n x^n,
\]

where \( m \) is a fixed nonnegative integer, \( a_m \neq 0 \), it is perhaps more convenient to proceed as follows. We have \( d_0 = 1 \), and for \( n > 0 \) we have, by Theorem 4.2,

\[
d_n = \sum_{j=1}^{n} (-1)^j a_{k_1 + m} \cdots a_{k_j + m}/(a_m)^j,
\]

where the sum is over all compositions \( k_1 + \cdots + k_j = n \), each \( k_i \geq 1 \). For \( t > 0 \) define \( G_{t,n}(s) \) and \( G(t; n, j) \) by means of

\[
\sum_{n=0}^{\infty} G_{t,n}(s) \frac{x^n}{n!} = \exp \left( s \sum_{r=t+1}^{\infty} a_r x^r \right),
\]

\[
G_{t,n}(s) = \sum_{j=1}^{[n/(t+1)]} G(t; n, j)x^j.
\]

Then

\[
j! G(t; n, j) = \sum n! a_{k_1} \cdots a_{k_j},
\]

where the sum is over all compositions \( k_1 + \cdots + k_j = n \), each \( k_i \geq t + 1 \). The number \( G(t; n, j) \) has the same interpretation as \( F(n, j) \), except each block used in a set partition of \( \{1, \ldots, n\} \) must contain at least \( t + 1 \) elements. For example, \( G(1; 4, 2) = 12a_2^2 \) and \( G(2; 4, 2) = 0 \). By (4.7) and (4.10) we have

\[
d_n = \sum_{j=1}^{n} (-1)^j j! (a_m)^{-j} G(m; n + mj, j)/(n + mj)!. \tag{4.11}
\]

By using the principle of inclusion-exclusion and the identity

\[
\sum_{j=r}^{n} \binom{j}{r} = \binom{n+1}{r+1}
\]

(see also the derivation of formula 18 in [12, p. 598]), we can derive the formula

\[
d_n = \sum_{j=1}^{n} (-1)^j j! (a_m)^{-j} \binom{n+1}{j+1} G(m-1; n + mj, j)/(n + mj)!. \tag{4.12}
\]

So if \( c_n \) is defined by (4.1) and \( d_n \) by (4.6), it is always possible to write
"explicit" formulas for $c_n$ and $d_n$, as shown by Theorem 4.2 and (4.7). It is also possible to write $c_n$ and $d_n$ as linear combinations of numbers which have a combinatorial interpretation, as shown by Theorem 4.3, (4.11) and (4.12). The next theorem shows it is always possible to find an application for the numbers $c_n$ and $d_n$ (see [12, pp. 587–599]).

**Theorem 4.4.** If $c_n$ is defined by (4.1) and $f(x)$, $h(x)$ are functions defined for positive integers $x$, then

\begin{equation}
(4.13) \quad h(n) = \sum_{i=0}^{n-1} a_i f(n - i)
\end{equation}

if and only if

\begin{equation}
(4.14) \quad f(n) = \sum_{m=0}^{n-1} c_m h(n - m).
\end{equation}

**Proof.** Suppose (4.13) holds. Then

\[
\sum_{n=1}^{\infty} h(n)x^{n-1} = \sum_{n=1}^{\infty} x^{n-1} \sum_{i=0}^{n-1} a_i f(n - i)
\]

\[
= \sum_{i=0}^{\infty} a_i x^i \sum_{n=i+1}^{\infty} f(n - i)x^{n-i-1}
\]

\[
= \left(\sum_{i=0}^{\infty} c_i x^i\right)^{-1} \sum_{n=i+1}^{\infty} f(n - i)x^{n-i-1}.
\]

This implies

\[
\left(\sum_{i=0}^{\infty} c_i x^i\right)\left(\sum_{n=1}^{\infty} h(n)x^{n-1}\right) = \sum_{n=1}^{\infty} f(n)x^{n-1}
\]

and (4.14) follows. If we assume (4.14), we use a similar method to prove (4.13).

We note that several formulas in [12, pp. 219, 247, 599] involving the Bernoulli numbers are special cases of the theorems of this section.

5. Relationship of $A_n$ to the Stirling Numbers. We now apply the results of Section 4 to the numbers $A_n$. From (1.1) and (4.7) we have, for $n > 0$,

\begin{equation}
(5.1) \quad A_n = n! \sum_{j=1}^{n} \frac{(-2)^j}{(k_1 + 2)! \cdots (k_j + 2)!},
\end{equation}

the sum being over all compositions $k_1 + \cdots + k_j = n$, each $k_i > 1$. This can be compared to a similar formula for the Bernoulli numbers [12, p. 247]:

\[
B_n = n! \sum_{j=1}^{n} \frac{(-1)^j}{(k_1 + 1)! \cdots (k_j + 1)!}.
\]

To find formulas corresponding to (4.11) and (4.12), we define $b_{t, n}(s)$ and $b(t; n, j)$ by means of

\begin{equation}
(5.2) \quad \sum_{n=0}^{\infty} b_{t, n}(s) \frac{x^n}{n!} = \exp(s(e^x - 1 - \cdots - x^t/t!))
\end{equation}
and

\[ b_{r,n}(s) = \sum_{j=1}^{[n/t+1]} b(t; n, j)x^j. \]  

(5.3)

Then (5.2) and (5.3) imply

\[ (e^x - 1 - x - \cdots - x^j/j!)^j = \sum_{n=j}^{\infty} j! b(t; n, j)\frac{x^n}{n!}. \]  

(5.4)

Using a different notation, these definitions were made by Riordan [16, p. 102, problem 7]. The numbers \( b(0; n, j) \) are the Stirling numbers of the second kind, which are very important in combinatorial analysis and finite differences. See [12] and [16] for applications. We shall use the notation

\[ b(0; n, j) = S(n, j). \]  

(5.5)

The numbers \( b(1; n, j) \), called the associated Stirling numbers of the second kind, have also been studied [16, p. 77], [12, pp. 171-173], [3]. Following Riordan, we shall use the notation

\[ b(1; n, j) = b(n, j). \]  

(5.6)

We shall also write

\[ b(2; n, j) = g(n, j). \]  

(5.7)

The numbers \( b(t; n, j) \) have the following interpretations (see the remarks following Theorem 4.3): \( b(t; n, j) \) is the number of set partitions of \( \{1, \ldots, n\} \) consisting of exactly \( j \) blocks, where each block contains at least \( t + 1 \) elements. Another interpretation is that \( b(t; n, j) \) is the number of ways of placing \( n \) distinct objects into \( j \) nondistinct cells, where each cell must contain at least \( t + 1 \) objects.

By (4.11) and (4.12), we have the following formulas:

\[ A_n = \sum_{j=1}^{n} (-1)^j \binom{n + 2j}{n}^{-1} \left[ 1 \cdot 3 \cdots (2j - 1) \right]^{-1} g(n + 2j, j), \]  

(5.8)

\[ A_n = \sum_{j=1}^{n} (-1)^j \binom{n + 1}{j + 1} \binom{n + 2j}{n}^{-1} \left[ 1 \cdot 3 \cdots (2j - 1) \right]^{-1} b(n + 2j, j). \]  

(5.9)

We can compare (5.8) and (5.9) to similar formulas for the Bernoulli numbers [12, pp. 219, 599]. Since [16, p. 77]

\[ b(n, j) = \sum_{k=0}^{j} (-1)^k \binom{n}{k} S(n - k, j - k), \]

we have, from (5.9),

\[ A_n = \sum_{j=1}^{n} \sum_{k=1}^{j} (-1)^k \binom{n + 1}{j + 1} \binom{n + 2j}{n} \binom{n + 2j}{j - k}^{-1} \left[ 1 \cdot 3 \cdots (2j - 1) \right]^{-1} S(n + j + k, k). \]  

(5.10)

The integers \( g(n, j) \) defined by (5.4) and (5.7) have properties similar to those of the Stirling numbers and associated Stirling numbers of the second kind. In particular, with \( g(0, 0) = 1 \), we have
NUMBERS GENERATED BY THE RECIPROCAL OF $e^x - x - 1$

(5.11) $$g(n + 1, j) = jg(n, j) + \binom{n}{2} g(n - 2, j - 1),$$

and we can easily compute a few values of $g(n, j)$:

<table>
<thead>
<tr>
<th>$n$</th>
<th>$j$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
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<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>35</td>
<td>91</td>
<td>210</td>
<td>456</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>280</td>
</tr>
</tbody>
</table>

We also have

(5.12) $$g(n, j) = \sum_{k=0}^{j} (-1)^k \binom{n}{2k} [1 \cdot 3 \cdots 2k - 1] b(n - 2k, j - k),$$

(5.13) $$b(n, j) = \sum_{k=0}^{j} \binom{n}{2k} [1 \cdot 3 \cdots 2k - 1] g(n - 2k, j - k).$$

Formulas (5.11), (5.12) and (5.13) can be proved in a more general setting. Following Riordan [16, pp. 76—78], we see that

(5.14) $$b_{t,n+1}(s) = \sum_{r=0}^{n-t} \binom{n}{r} b_{t,n}(s),$$

(5.15) $$b_{t,n}(s) = \sum_{r=0}^{n} \frac{n!(t!)^r(-s)^r}{r!(n-tr)!} b_{t-1,n-tr}(s),$$

(5.16) $$b_{t,n-1}(s) = \sum_{r=0}^{n} \frac{n!(t!)^r(-s)^r}{r!(n-tr)!} b_{t,n-tr}(s).$$

By differentiating (5.2) with respect to $u$ and subtracting $s$ times the derivative of (5.2) with respect to $s$, we derive

(5.17) $$b(t; n + 1, j) = jb(t; n, j) + \binom{n}{t} b(t; n - t, j - 1),$$

with $b(t; 0, 0) = 1$. Also, from (5.2) and (5.3),

(5.18) $$b(t; n, j) = \sum_{j_1 \geq t+1} \frac{n!}{j_1! \cdots k_j!} ,$$

the sum being over all compositions $k_1 + \cdots + k_j = n$, each $k_i \geq t+1$.

A natural generalization of (1.1) is

(5.19) $$e^{-x} \frac{x^m}{m!} = \sum_{n=0}^{\infty} A_{m,n} \frac{x^n}{n!} .$$

Definition (5.19) was made in [8], and arithmetic properties of the rational numbers $A_{m,n}$ were discussed in that paper. It follows that

(5.20) $$A_{m,n} = \sum_{j=1}^{n} (-m!)^j/j! n! b(m; n + mj, j)/(n + mj)! .$$
and

\[(5.21) \quad A_{m,n} = n! \sum_{j=1}^{n} \frac{(-m)!}{(m + k_1)! \cdots (m + k_j)!},\]

the sum being over all compositions \(k_1 + \cdots + k_j = n\), each \(k_i \geq 1\). Applying Theorem 4.4, we see that if

\[
m - \epsilon(n) > \frac{m}{2},
\]

then

\[n - 1 \sum_{x=1}^{n-1} x^j = 0 \quad x \geq 1\]

From (5.19) we have \(A_{1,n} = B_n\) and \(A_{2,n} = A_n\).

6. Miscellaneous Results. From (1.1) and (2.3) we see that

\[(6.1) \quad (e - 2)^{-1} = 2 \sum_{n=0}^{\infty} A_n/n!;\]

and the convergence appears to be very rapid since

\[
(e - 2)^{-1} = 1.392211191 \cdots \quad \text{and} \quad 2 \sum_{n=0}^{5} A_n/n! = 1.392210464 \cdots.
\]

By letting \(x = -1\) in (1.1), we have

\[(6.2) \quad e = 2 \sum_{n=0}^{\infty} (-1)^n A_n/n!,\]

and again the convergence is rapid. More generally, from (1.5) we have for all \(z\)

\[(6.3) \quad e^{1-z} = 2 \sum_{n=0}^{\infty} (-1)^n A_n(z)/n!.\]

We can compare the sizes of \(A_n\) and the Bernoulli numbers. From (2.1) and (2.2) we see that

\[(6.4) \quad |A_n| < n! \sum_{s=1}^{\infty} (2\pi s)^{-n},\]

and since [12, p. 244]

\[2(n!) \sum_{s=1}^{\infty} (2\pi s)^{-n} = |B_n|,
\]

for \(n\) even, we see that for \(n = 2m, m > 0,\)

\[(6.5) \quad 2|A_{2m}| < |B_{2m}|,
\]

and it follows [12, p. 245] that for \(m > 0\)
(6.6) \[ 24|A_{2m}| < (2m)! (2\pi)^{2m}. \]

Generally, using the approximation

\[ |A_n| = n! (\cos n\theta) r_1^{-n}, \]

we conjecture that for all \( n > 0 \)

(6.7) \[ |A_n| < n! 7^{-n}. \]

It was proved in [8] that the numbers \( A_n \) are not bounded.

As we saw in Section 3, there is still a question of whether or not \( A_n(x) \) can have rational roots when \( n \) is odd. The following theorems shed a little light on this situation.

**Theorem 6.1.** If \( p \) is a prime number, \( p > 3 \), and if \( p - 2 \) divides \( n \), then \( A_n(x) \) has no rational roots.

**Proof.** By the proof of Theorem 6.2 in [6], we have

\[ p^m A_m(p-2)(u/3) \equiv p^m A_m(p-2) \equiv 0 \pmod{p}. \]

It follows that \( u/3 \) cannot be a root of \( A_m(p-2)(x) \).

**Theorem 6.2.** Suppose \( u/3 \) is a rational root of \( A_n(x) \) and \( n = 1 + 3^t k, k \equiv 0 \pmod{3} \). If \( t = 1 \), then \( u \equiv 1 \pmod{9} \). If \( t > 1 \), then \( u \equiv 1 \pmod{3^t+2} \).

**Proof.** We know from Theorem 3.1 that \( u = n \equiv 1 \pmod{3} \). Note that

\[ \binom{n}{r} 3^r A_r = n(n-1) \cdots (n-r+1) 3^r A_r/r!, \]

so

\[ \sum_{r=3m+2}^{n} \binom{n}{r} 3^{r-1} A_r u^{n-r} \equiv 0 \pmod{3^t+m-1}. \]

From (3.6) we have

\[ 0 \equiv \sum_{r=0}^{4} \binom{n}{r} 3^{r-1} A_r u^{n-r} \]

\[ \equiv u^{n-1}(u-1)/3 + 3^{t-1} ku^{n-4}(-1 - 2u + 10u^2 - 40u^3) / 40 \]

\[ \equiv u^{n-1}(u-1)/3 \pmod{3^t}, \]

which implies \( u \equiv 1 \pmod{3^{t+1}} \). In fact, if \( t > 1 \),

\[ 0 \equiv u^{n-1}(u-1)/3 - 3^t k \cdot 11/40 - 3^t k \cdot 47/1400 \pmod{3^{t+1}}, \]

which implies \( u \equiv 1 \pmod{3^{t+2}} \).

We can use the method of Theorem 6.2 to get more information about \( u \), if \( u/b \) is a rational root of \( A_n(x) \). Suppose \( n = 1 + 3^t k, t > 2, k \equiv 0 \pmod{3} \) and suppose \( u = 1 + 3^{t+2} m \). Then we have
\[ 0 \equiv \sum_{r=0}^{10} \binom{n}{r} 3^{r-1} u^{n-r} \equiv 3^{r+1} m + 3^{r} k(-11/40 - 47/1400) + 3^{r+1} k(5120) \]

\[ \equiv 3^{r+1} m - 3^{r+1} k \pmod{3^{r+2}}. \]

For \( r = 8, 9, 10 \) we have used (1.3). Thus we see that in this case we must have \( m \equiv k \pmod{3} \).

If \( n = 4 + 9k \) or \( 7 + 9k, k \neq 0 \pmod{3} \), we can use this method to show that \( u \equiv 19 \pmod{27} \). If \( n = 1 + 9k, k \neq 0 \pmod{3} \), we can use this method to show that \( u \equiv 1 \pmod{243} \).

By these results and the remarks following Theorem 3.2, we see that if \( A_{n}(1/3) = 0 \), then \( n \equiv 1 \pmod{36} \).

Returning to definitions (5.2) and (5.3), we can find a relationship between \( b_{2,n}(s) \) and the Hermite polynomials. Let

\[ g_{n}(s) = b_{2,n}(s), \quad a_{n}(s) = b_{0,n}(s). \]

From (5.2) we have

\[ \sum_{n=0}^{\infty} g_{n}(s) \frac{u^{n}}{n!} = \exp[s(e^{u} - 1)] \exp[-s(u + u^{2}/2)]. \]

In [11, p. 181] the Hermite polynomial \( H_{n}(x) \) is defined by means of

\[ \exp(xu - u^{2}/2) = \sum_{n=0}^{\infty} H_{n}(x) \frac{u^{n}}{n!}. \]

Thus by (6.8) and (6.9) we have

\[ g_{n}(1) = \sum_{r=0}^{n} \binom{n}{r} a_{r}(1) H_{n-r}(-1), \]

where \( H_{0}(-1) = 1, H_{1}(-1) = -1 \) and

\[ H_{n+1}(-1) = -H_{n}(-1) - nH_{n-1}(-1). \]

It follows that

\[ g_{n}(1) = \sum_{r=0}^{n} \sum_{j=0}^{r} \binom{n}{r} S(n-r, j) H_{r}(-1). \]

The number \( g_{n}(1) \) is the number of ways of putting \( n \) different objects into \( n \) like cells, where each nonempty cell must contain at least three objects.

We conclude with two tables. Table 1 gives the value of \( n\theta_{1} \) (modulo \( 360^{\circ} \)), rounded off to the nearest degree, and also the values of \( \cos n\theta_{1} \) rounded off at the third place. This is done for \( 1 \leq n \leq 46 \). Table 2 indicates when the pattern of Theorem 2.2 changes for \( A_{n} \) when \( n = 46k + s \).
Table 1

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7. Acknowledgement. I would like to thank Professor L. Carlitz for his advice and encouragement. I would also like to thank Professor J. W. Sawyer of the Wake Forest computer center and H. R. P. Ferguson of Brigham Young University, for confirming approximations (2.3) with the following computations:

\[ a = 2.0888430156130, \]
\[ b = 7.46148928565425, \]
\[ \theta_1 = 74.36041657449774°, \]
\[ r_1 = 7.74836031065984. \]
Department of Mathematics  
Wake Forest University  
Winston-Salem, North Carolina 27109


5. L. CARLITZ, "Set partitions," *Fibonacci Quart.* (To appear.)


