Application of Method of Collocation on Lines for Solving Nonlinear Hyperbolic Problems

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Abstract. A collocation on lines procedure based on piecewise polynomials is applied to initial/boundary value problems for nonlinear hyperbolic partial differential equations. Optimal order a priori estimates are obtained for the error of approximation. The Crank-Nicholson discretization in time is studied and convergence rates of the collocation-Crank-Nicholson procedure are established. Finally, the superconvergence is verified at particular points for linear hyperbolic problems.

Introduction. We consider the nonlinear hyperbolic problem

\[ p(x, t, u)D_x^2u - q(x, t, u)D_x^2u = f(x, t, u, D_xu), \quad (x, t) \in (0, 1) \times (0, T], \]

subject to the initial conditions

\[ u(x, 0) = u_0, \quad D_tu(x, 0) = u_1, \quad x \in (0, 1), \]

and to Dirichlet boundary conditions for \( t > 0 \). We examine the convergence of the collocation on lines procedure using piecewise polynomials with continuous first derivatives as the approximating functions.

In Section 4 we obtain optimal-order asymptotic estimates for the error of the approximation in the \( L_\infty \)-norm. In Section 5, the Crank-Nicholson discretization of the resulting system of ordinary differential equations is studied and convergence rates of the collocation on lines–Crank-Nicholson procedure are established. Finally, in Section 6 the superconvergence phenomenon is established locally for a linear hyperbolic problem.

The method of collocation on lines was proposed first by Kantorovich [7]. The convergence of this method for a problem of mathematical physics was investigated by E. B. Karpilovskaya [8]. Yartsev [11], [10] proved convergence for linear elliptic and biharmonic type problems using trigonometric polynomials as basis functions. Douglas and Dupont [3], have studied the same method using piecewise cubic Hermite polynomials for a nonlinear parabolic problem and in [4] verified the superconvergence locally for the heat equation. Finally, Douglas and Dupont [5] generalized and extended their results in [3], [4]. The results in this paper are from the author’s thesis [6].

1. Preliminary Results. Let \( \Delta_x = (x_i)_i^{N} \) be a partition of \([0, 1]\), \( I = [0, 1] \), \( h_j = x_{i+1} - x_i \), \( I_j = [x_j, x_{j+1}] \) and \( h = \max_j |x_{j+1} - x_j| \). Throughout this paper...
we denote by $P_r$ the set of polynomials of degree less than $r$ and $P_r,\Delta_x$ the set of functions that are polynomials of degree $r - 1$ in each subinterval $[x_i, x_{i+1}]$. We take $-1 < \rho_1 < \rho_2 < \cdots < \rho_k < 1$ and $w_j > 0, j = 1, \ldots, k$, to be Gaussian points and weights, respectively, so that

$$\int_{-1}^{+1} p(x)dx = \sum_{i=1}^{k} p(\rho_i)w_i, \quad p \in P_{2k}([-1, 1]).$$

The Gaussian points and weights in the subinterval $[x_i, x_{i+1}]$ are

$$\xi_{ki+i} = (x_i + x_{i+1})/2 + \rho_i h_j/2, \quad w_i = h_j w_j/2, \quad i = 1, \ldots, k.$$

We introduce two pseudo-inner products corresponding to Gaussian quadrature and composite Gaussian quadrature:

$$(f, g)_{h_i} \equiv \frac{h_i}{2} \sum_{j=1}^{k} w_i f(\xi_{ki+i}) \cdot g(\xi_{ki+i}),$$

and

$$(f, g)_h \equiv \sum_{j=0}^{N-1} (f, g)_{h_j},$$

with

$$|f|_h \equiv \sum_{j=0}^{N-1} (f, f)_{h_j}.$$ 

For later use, we state without proof the lemmas:

**Lemma 1.1.** The seminorm $|f|_h$ is positive definite for all $f \in P_{k+2,\Delta_x} \cap C^1[0, 1]$ with $f(0) = f(1) = 0$.

**Lemma 1.2.** If $f, g \in P_{k+2,\Delta_x} \cap C^1[0, 1]$, then

$$-(D_x^2 f, g)_h = (D_x f, D_x g) - D_x f \cdot g|_0$$

$$+ \frac{(k + 1)k}{(2k)!} \sum_{j} \frac{D_x^{k+1} f_j}{(k + 1)!} \cdot \frac{D_x^{k+1} g_j}{(k + 1)!} \int_{x_j}^{x_{j+1}} \prod_{i=1}^{k} (x - \xi_{ki+i})^2 dx.$$ 

**Lemma 1.3.** If $f \in \{v \in P_{k+2,\Delta_x} \cap C^1, v(0) = v(1) = 0\}$, then

$$(D_x f, D_x f) \leq - (D_x^2 f, f)_h \leq 2(D_x f, D_x f)$$

and

$$|D_x f|_h^2 \leq (D_x f, D_x f).$$

**Lemma 1.4.** If $f \in P_{k+2,\Delta_x} \cap C^1[0, 1]$, then

$$|f|_h \leq \lambda \|f\|_{L^2(I)},$$

where $\lambda$ is the maximum eigenvalue of the matrix $A_{k+1} \equiv [\Sigma_{i=1}^{k} w_i L_i(\rho_i)L_j(\rho_j)]$ and $L_i$ denotes the ith degree Legendre polynomials in $[-1, 1]$.

Let $H^k$ be the Sobolev space of functions having $L^2$-derivatives of order $k$ on $I$ and $H^k_0 \equiv \{u \in H^k|u(0) = u(1) = 0\}$. 

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Lemma 1.5. For \( f \in H^1 \) we have

\[
(D_x f, D_x f) + |f|_h^2 \geq \frac{1}{4} \| f \|_{H^1(I)}^2.
\]

The above lemmas are established in [6], proofs also appear in [5]. Lemmas 1.2, 1.3 and 1.5 have been first proved for the case of cubic Hermite polynomials by Douglas and Dupont [3].

2. Approximation Theory. In [6] we show that \( R_k(x) \equiv D^k_x(1 - x^2)^{k+2}, k = 0, 1, \ldots, \) on \((-1, 1)\) are orthogonal polynomials. By Rodrigues' formula we see that \( D^2_x R_k(x) = D^{k+2}_x(1 - x^2)^{k+2} \) is a multiple of the Legendre polynomial on the interval \((-1, 1)\). We now establish some properties of these polynomials.

Lemma 2.1. If \( k \geq 3 \),

\[
(D^\mu_x R_{k-2}, x^\nu)_h = 0, \quad \mu = 0, 1, 2, \nu < \mu.
\]

Proof. Since \( D^\mu_x R_{k-2} x^\nu \) is a polynomial of degree \( K + 2 - \mu + \nu \), we have for \( k \geq 3 \),

\[
(D^\mu_x R_{k-2}, x^\nu)_h = \int_{-1}^{1} D^\nu_x R_{k-2} x^\nu dx.
\]

Lemma 2.1 now follows by using integration by parts and the fact that \( D^\mu_x R_{k-2} \) vanishes at \( x = \pm 1 \) and \( D^2_x R_{k-2} \) vanishes at the Gaussian points. Note that for \( k \geq 2 \),

\[
(D_x R_k, 1)_h = (D^2_x R_{k-2}, x^\nu)_h = 0.
\]

We define an interpolation operator

\[
T_h: C^1(I) \rightarrow \mathbb{P}_{k+2, \Delta_x} \cap C^1(I)
\]

such that

\[
(T_h v)(x_i) = v(x_i),
\]

\[
(D_x T_h v)(x_i) = (D_x v)(x_i), \quad l = 0, 1, \ldots, N,
\]

\[
(T_h v)(\tau_{i,j}) = v(\tau_{i,j}), \quad i = 1, \ldots, k, j = 1, \ldots, N,
\]

where \( \tau_{i,j} \equiv x_j + a_i (x_{j+1} - x_j) \) and the \( a_i \)'s are the roots in the interval \((0, 1)\) of the orthogonal polynomials \( R_{k-2}(x) \).

Lemma 2.2. Assume that \( u \in H^{k+4}(I) \) and let \( e = u - T_h u \). Then there is a constant \( K \) independent of \( h \) so that

\[
|D^l_x e|_h \leq Kh^{k-l+2}\|u\|_{H^{k+2}(I)}, \quad l = 0, 1,
\]

\[
|D^2_x e|_h \leq Kh^{k-l+1}\|u\|_{H^{k+3}(I)}
\]

\[
|(D_x e, 1)_h| \leq K h^{2k+5/2}\|u\|_{H^{k+3}(I)},
\]

\[
|(D^2_x e, 1)_h| \leq K h^{2k+5/2}\|u\|_{H^{k+4}(I)}.
\]

Proof. It follows easily from Lemma 2.1 and Peano's Kernel Theorem [9].
3. Collocation on Lines. In this section we consider the problem of approximating the solution of the nonlinear hyperbolic equation

\[(3.1)\quad p(x, t, u)D_x^2u - q(x, t, u)D_xu = f(x, t, u, Dxu), \quad (x, t) \in (0, 1) \times (0, T],\]

subject to the initial conditions

\[(3.2)\quad u(x, 0) = \alpha_1(x), \quad D_xu(x, 0) = \alpha_2(x), \quad 0 < x < 1,\]

and the boundary conditions

\[(3.3)\quad u(0, t) = 0, \quad u(1, t) = 0, \quad 0 < t \leq T.\]

Assume that the coefficients satisfy

\[(3.4)\quad 0 < c_1 \leq p(x, t, u) \leq C_1, \quad c_2 \leq q(x, t, u) \leq C_2,
\]

for \(0 \leq x \leq 1, 0 \leq t \leq T\) and \(-\infty < u < +\infty\). Also, we assume that \(p, q, f\) are continuously differentiable functions of their arguments and uniformly bounded.

Throughout, we assume that this problem has a solution, \(u\).

Let \(S_{\Delta x} = P_{k+2,\Delta x} \cap C^1\) and \(S_{\Delta x} \cap H^1_0\) be spanned by the basis functions \(\{B_i\}_{i=1}^{kN}\). We seek an approximation \(u_h(x, t)\) to \(u\) of the form

\[
\begin{align*}
   u_h(x, t) = \sum_{i=1}^{kN} \beta_i(t)B_i(x).
\end{align*}
\]

The coefficients \(\{\beta_i(t)\}_{i=1}^{kN}\) as functions of time are the solutions of the nonlinear ordinary differential equations

\[(3.5)\quad (p(u_h)D_x^2u_h - q(u_h)D_xu_h - f(u_h, D_xu_h))(\xi_i, t) = 0, \quad 0 < t \leq T, i = 1, \ldots, kN,\]

and

\[(3.6)\quad u_h(\xi_i, 0) = \hat{\alpha}_1(\xi_i), \quad D_xu_h(\xi_i, 0) = \hat{\alpha}_2(\xi_i), \quad k = 1, \ldots, kN,\]

where \(\hat{\alpha}_1, \hat{\alpha}_2\) are the \(S_{\Delta x}\)-interpolants of \(\alpha_1(x), \alpha_2(x)\) respectively.

Although these are the equations which one solves in practice, the analysis is more conveniently made if one considers the equivalent problem of finding \(u_h \in S_{\Delta x} \cap H^1_0\) such that

\[(3.7)\quad (p(u_h)D_x^2u_h - q(u_h)D_xu_h - f(u_h, D_xu_h), B_i)_h = 0, \quad 0 < t \leq T, i = 1, \ldots, kN,\]

and

\[(3.8)\quad u_h(\xi_i, 0) = \hat{\alpha}_1(\xi_i), \quad D_xu_h(\xi_i, 0) = \hat{\alpha}_2(\xi_i), \quad i = 1, \ldots, kN.\]

**Lemma 3.1.** The collocation method (3.5), (3.6) and the discrete Galerkin method (3.7), (3.8) each possess a unique solution for \(0 < t \leq T\). Moreover, these solutions are identical if the processes are started from the same initial values.

**Proof.** It follows from Lemma 4.1 in [5].
4. Error Analysis. In this section, we find a priori error bounds for the collocation on lines procedure. We consider the problem of finding $u_h \in S_{\Delta x} \cap H_0^1$ such that

\[(4.1) \quad (p(u_h)D_t^2 u_h - D_x^2 u_h - f(u_h, D_x u_h), v)_h = 0, \quad 0 < t \leq T,\]

for all $v \in S_{\Delta x} \cap H_0^1$.

In order to find estimates for the error $u - u_h$ in the $L_\infty$-norm, we assume that $u(\cdot, t) \in C^1(I)$ and define $w(\cdot, t) \equiv T_h u$ which is in $S_{\Delta x}$. Then we find a priori bounds for the difference $w - u_h \in S_{\Delta x}$; and applying known approximation results to the difference $u - w$, we obtain bounds for the error of the collocation on lines procedure.

If $X$ is a normed space and $\psi: [0, T] \rightarrow X$, define

\[\|\psi\|_{L^2(0,T;X)} = \int_0^T \|\psi(t)\|^2_X \, dt, \quad \|\psi\|_{L^\infty(0,T;X)} = \sup_{0 < t < T} \|\psi(t)\|_X.\]

**Theorem 4.1.** If

(i) the coefficients in (3.1) have bounded third derivatives and satisfy conditions (3.4),

(ii) $u \in L^\infty(0, T; H^{k+4})$, $D_t u \in L^2(0, T; H^{k+4})$ and $D_t^2 u \in L^2(0, T; H^{k+4})$, respectively, then for the error of approximation we have

\[\|u - u_h\|_{L^\infty(0,T;L^\infty)} \leq K [\|u\|_{L^\infty(0,T;H^{k+4}(I))} + \|D_t^2 u\|_{L^2(0,T;H^{k+4}(I))} + \|D_t^2 u_h\|_{L^2(0,T;H^{k+4}(I))}] h^{k+2},\]

where $K$ is a constant independent of $h$ and $u$.

**Proof.** Let $\eta \equiv u - w$ and $\xi \equiv u - u_h$. Then (3.1), (4.1) imply that

\[(p(u_h)D_t^2 \xi - D_x^2 \xi, v)_h = (p(u)D_t^2 w - p(w)D_t^2 \eta - p^2 u D_x^2 u, v)_h + (D_t^2 \eta, v)_h + (f(w, D_x u) - f(w, D_x w), v)_h + (f_t^1 \eta + f_t^2 \xi + f_{D_x}^3 D_x \xi, v)_h.\]

In (4.2) we choose $v = D_t \xi$ and in [6] we show that

\[\frac{1}{2} [\sqrt{p(u_h)} D_t \xi_h^2 + |\xi|^2_h - (D_x^2 \xi, \xi)_h] \leq K \int_0^T (|\xi|^2_h - D_x \xi_h^2) \, dt + \int_0^T (|\eta|^2_h + |D_t^2 \eta|^2_h) \, dt + K \int_0^T |D_t \xi|^2_h \, dt.

(4.3)

\[+ K \{ |\xi|^2_h(0) - (D_x^2 \xi, \xi)_h(0) + |\sqrt{p(u_h)} D_t \xi|^2_h(0) \} + \int_0^T (D_t^2 \eta, D_t \xi)_h \, dt + \int_0^T (f(w, D_x u) - f(w, D_x w), D_t \xi)_h \, dt.\]

Integration by parts gives

\[\int_0^T (D_t^2 \eta, D_t \xi)_h \, dt = (D_t^2 \eta, \xi)_h|_0^T - \int_0^T (D_t D_x^2 \eta, \xi)_h \, dt,\]
and

\[
\int_0^t (f(w, D_x u) - f(w, D_x w), D_t \xi)_h \, d\tau
= (f(w, D_x u) - f(w, D_x w), \xi)_h \bigg|_0^t - \int_0^t (D_f (f(w, D_x u) - f(w, D_x w)), \xi)_h \, d\tau.
\]

Using Poincaré's inequality, the elementary inequality \(|cd| \leq (4p)c^2 + pd^2\) and Lemma 2.2 in [6] we have obtained

\[
\left| \int_0^t (D_t D_x^2 \eta, \xi)_h \, d\tau \right| \leq \frac{1}{16} \int_0^t \left[ -\langle D_x^2 \xi, \xi \rangle_h + \| \xi \|^2 \right] \, d\tau
+ K \sum_{i=1}^N h_j^{2k+4} \int_0^t \| D_t u(\cdot, \tau) \|^2_{H^{k+3}(I_j)} \, d\tau.
\]

Using Taylor's theorem, we can easily show that

\[
(f(w, D_x u) - f(w, D_x w), w - u_h)_h
= \sum_{j=1}^N (f_{D_x^j} (w, D_x w))_{x = \epsilon_{k(j-1)}+1} D_x \eta + \omega h_j D_x \eta, \xi)_h,
\]

where \(\omega\) is bounded independent of \(h_j\). It follows from Lemma 2.2

\[
|f_{D_x^j} (w, D_x w)|_{x = \epsilon_{k(j-1)}+1} D_x \eta + \omega h_j D_x \eta, \xi)_h \leq K h_j^{k+3} \| u(\cdot, \tau) \|_{H^{k+3}(I_j)} \| D_x \xi(\cdot, \tau) \|_{L^2(I_j)},
\]

and

\[
|\omega h_j D_x \eta, \xi)_h | \leq K h_j^{k+2} \| u(\cdot, \tau) \|_{H^{k+2}(I_j)} \| \xi \|_{H^{k+2}(I_j)}.
\]

Moreover, we obtain

\[
|f(w, D_x u) - f(w, D_x w), \xi)_h |
\leq \frac{1}{16} \left[ -\langle D_x^2 \xi, \xi \rangle_h + \| \xi \|^2 \right] + K \sum_{j=1}^N h_j^{2k+4} \| u(\cdot, \tau) \|^2_{H^{k+3}(I_j)}.
\]

Following similar arguments as above, we show that

\[
\int_0^t (D_t \{ f(w, D_x u) - f(w, D_x w) \}, \xi)_h \, d\tau
\leq \frac{1}{16} \int_0^t \left[ -\langle D_x^2 \xi, \xi \rangle_h + \| \xi \|^2 \right] \, d\tau
+ K \sum_{j=1}^N h_j^{2k+4} \int_0^t \| u(\cdot, \tau) \|^2_{H^{k+3}(I_j)} \, d\tau
+ \| D_t u(\cdot, \tau) \|^2_{H^{k+3}(I_j)} \, d\tau.
\]

It follows from (4.3)-(4.5), (1.3) and Gronwall's Lemma [6] that
\[
\|w - u_h\|_{L^\infty(0,T;L^\infty)}^2 \leq K \left[ \|w - u_h\|_{H^1(I)}^2 + \|D_t (w - u_h)\|_{L^2(I)}^2 \right] \\
+ \sum_{j=1}^{N} h_j^{2k+4} \left( \|u\|_{L^\infty(0,T;H^{k+4}(I_j))}^2 + \|D_t u\|_{L^2(0,T;H^{k+4}(I_j))}^2 + \|D^2_t u\|_{L^2(0,T;H^{k+4}(I_j))}^2 \right).
\]

(4.6)

It is an elementary consequence of Peano's Kernel Theorem that

\[
\|u - w\|_{L^\infty(0,T;L^\infty)}^2 \leq K \sum_{j=1}^{N} h_j^{2k+4} \|u\|_{L^\infty(0,T;H^{k+4}(I_j))}^2.
\]

(4.7)

Finally, from (4.6), (4.7) and assumption (iii) it follows that

\[
\|u - u_h\|_{L^\infty(I)} \leq Kh^{k+2} \left[ \|u\|_{L^\infty(H^{k+4}(I))} + \|D_t u\|_{L^2(H^{k+4}(I))} + \|D^2_t u\|_{L^2(H^{k+4}(I))} \right].
\]

This concludes the proof of Theorem 4.1.

5. Computational Considerations. In this section, we discuss the question of actually solving the system of ordinary differential equations (3.5), (3.6).

Let

\[
u^t_i \equiv u^t_i(x) = u^t_i(x, t^i), \quad t^i \equiv j \Delta t, \quad \Delta t = T/N,
\]

(5.1)

\[
u^{j+1/2} = \frac{1}{2} \left( \nu^t + \nu^t + \nu^t \right), \quad \nu^{j+1} = \frac{1}{4} \nu^{j+1} + \frac{1}{2} \nu^j + \frac{1}{4} \nu^j,
\]

\[
\partial_t \nu^{j+1} = \frac{1}{2} \left( \nu^{j+1} - \nu^j \right)/\Delta t, \quad \partial^2_t \nu^j = \frac{1}{2} \left( \nu^{j+1} - 2\nu^j + \nu^{j-1} \right)/\Delta t^2.
\]

Then the Crank-Nicholson-Collocation approximation \{u^t_i\}_{i=0}^{N} is defined such that

(i) \( (p(t^i, u^{j+1/2}_h) \partial^2_t u^t_i - q(t^i, u^{1/2}_h) D_x^2 u^t_i, v)_h = (f(t^i, u^{1/2}_h, D_x u^{1/2}_h), v)_h, \)

(5.2)

\( i = 1, \ldots, kN, j = 0, \ldots, N - 1, \)

(ii) \( u^t_0(0) = u^t_N(1) = 0, \quad j = 0, \ldots, N. \)

At the end of this section we discuss the choice of \( u^t_0, u^t_1 \). In order to analyze the convergence of the solution of (5.2) we consider the equivalent to (5.2) normalized problem

\[
(p(t^i, u^{j+1/2}_h) \partial^2_t u^t_i, v)_h - (D_x^2 u^{j+1/2}_h, v)_h = (f(t^i, u^{1/2}_h, D_x u^{1/2}_h), v)_h, \quad v \in S_{\Delta x} \cap H^1_0, 0 \leq j < N.
\]

(5.3)

Also, we introduce the notation

\[
\|u\|_{L^2_{\Delta t}(0,T;X)}^2 \equiv \sum_{0 \leq t \leq T} \|u^t\|_{X}^2 \Delta t,
\]

\[
\|u\|_{L^\infty_{\Delta t}(0,T;X)}^2 \equiv \max_{0 \leq t \leq T} \|u^t\|_{X}^2.
\]
Theorem 5.1. Assume the hypotheses (i), (ii) of Theorem 4.1 hold. Further, assume \( D_x^3 u, D_x^4 u \) are in \( L^\infty(0, T; L^2(I)) \) and
\[
\|(u_n - w)\|_{H^1(I)} + \|\partial_x (u_n - w)\|_{L^2(I)} = O(h^{k+2}).
\]
For \( \Delta t \) sufficiently small there exists a unique solution of the Crank-Nicholson-Collocation equations (5.2) and for the error of approximation we have
\[
\|u - u_n\|_{L^\infty(0,T;L^\infty)} \leq C(h^{k+2} + (\Delta t)^2),
\]
where \( C \) depends on \( u \) and is independent of \( h, \Delta t \).

Proof. It is easily seen that a unique solution of (5.2) exists under assumption (i) and (3.3) for \( \Delta t \) sufficiently small. Throughout this proof we use the notation \( w = T_n u, \eta = u - w \) and \( \xi = u_n - w \). First, we observe that \( u \) satisfies
\[
(p(u^{i+1}, \eta^{i+1}) \partial_x^2 \xi^i, v) + (f(u^{i+1}, D_x u^{i+1}), v) + (e^i, v)
\]
for \( v \in S_{\Delta x} \cap H_x^1 \), where \( \|e^i\|_{L^2(I)} = O(\Delta t^2) ||D_x^4 u||_{L^2(I)} \).

After straightforward calculations and the application of the Mean Value Theorem, we obtain
\[
(p(u^{i+1}, \eta^{i+1}) \partial_x^2 \xi^i, v) - (D_x^2 \xi^i, v)
\]
\[
= (p\eta^{i+1} \partial_x^2 \eta^i, v) + (p\eta^{i-1} \partial_x^2 \eta^i, v) + (e^i, v) - (D_x^2 \eta^{i+1}, \eta^i v)
\]
\[
+ (f_{i+1} \eta^{i+1} + f_{i-1} \eta^{i-1} + f_{D_x u} D_x \xi^{i+1}, v)
\]
\[
+ (f(w^{i+1}, D_x w^{i+1}) - f(w^{i+1}, D_x u^{i+1}), v).
\]
In (5.5), we choose as test function \( v = (\xi^{i+1} - \xi^{i-1})/2t \) and then we obtain
\[
\frac{1}{2\Delta t} \left[ (|p(u^{i+1}) \partial_x^2 \xi^{i+1} h^2 + |\xi^{i+1} h^2 | - (D_x^2 \xi^{i+1}, \xi^{i+1})_h) \right]
\]
\[
- [\left| (|p(u^{i+1}) \partial_x^2 \xi^{i-1} h^2 + |\xi^{i-1} h^2 | - (D_x^2 \xi^{i-1}, \xi^{i-1})_h) \right] \}
\]
\[
\leq C(\xi^{i+1} h^2 + |\xi^{i-1} h^2 | + |\partial_x^2 \eta^{i+1} h^2 | + |\eta^{i+1} h^2 | + |\partial_x \eta^{i+1} h^2 | + |e^i h^2 |)
\]
\[
+ \left| \left( D_x^2 \eta^{i+1} , \frac{\xi^{i+1} - \xi^{i-1}}{2\Delta t} \right)_h \right|
\]
\[
+ \left| \left( f(w^{i+1}, D_x w^{i+1}) - f(w^{i+1}, D_x u^{i+1}), \frac{\xi^{i+1} - \xi^{i-1}}{2\Delta t} \right)_h \right|
\]
where \( C \) is a generic constant.
APPLICATION OF METHOD OF COLLOCATION

Following the same arguments as in Section 4 and using Lemma 2.2, we get

\[
\Delta t \sum_{j=1}^{n-1} \left( D_x^2 \eta_j \xi_j, \frac{\xi_j^{j+1} - \xi_j^{j-1}}{2\Delta t} \right)_h \\
\leq \frac{1}{\varepsilon} \left\{ - (D_x^2 \xi^{n-\frac{1}{2}}, \xi^{n-\frac{1}{2}})_h + |\xi^{n-\frac{1}{2}}|_h^2 - (D_x^2 \xi^{\frac{1}{2}}, \xi^{\frac{1}{2}})_h + |\xi^{\frac{1}{2}}|_h^2 \right\} \\
+ K \max_{0 < t' < T} \sum_{i=1}^{N-1} h_i^{2k+4} \|u_i\|_{H^{k+4}(I_i)} + \max_{0 < t' < T} \|e^t_i\|_{L^2(I)}^2 \\
(5.7) \\
+ \frac{1}{2\varepsilon} \Delta t \sum_{j=1}^{n-1} \left\{ - (D_x^2 \xi^{j+\frac{1}{2}}, \xi^{j+\frac{1}{2}})_h - (D_x^2 \xi^{j-\frac{1}{2}}, \xi^{j-\frac{1}{2}})_h \\
+ |\xi^{j+\frac{1}{2}}|_h^2 + |\xi^{j-\frac{1}{2}}|_h^2 \right\} \\
+ K \Delta t \sum_{j=1}^{n-1} \sum_{i=0}^{N-1} h_i^{2k+4} \|D_x^2 u_i\|_{H^{k+4}(I_i)}^2 \\
+ \Delta t \sum_{j=1}^{N-1} \|e^j_i\|_{L^2(I)}^2, \\
\}
\]

where \(\|e^j_i\|_{L^2(I)}^2 = O(\Delta t^2)\) for \(s = 1, 2\), \(K\) is a generic constant and \(\varepsilon\) a constant that can be small enough.

Finally, by arguments similar to those of Section 4 we can show that

\[
\Delta t \sum_{j=1}^{n-1} \left( f(w_j, D_x u_j, w_j, D_x w_j) - f(w_j, D_x w_j, w_j, D_x w_j), \frac{\xi^{j+\frac{1}{2}} - \xi^{j-\frac{1}{2}}}{\Delta t} \right)_h \\
\leq \frac{1}{\varepsilon} \left\{ - (D_x^2 \xi^{n-\frac{1}{2}}, \xi^{n-\frac{1}{2}})_h + |\xi^{n-\frac{1}{2}}|_h^2 - (D_x^2 \xi^{\frac{1}{2}}, \xi^{\frac{1}{2}})_h + |\xi^{\frac{1}{2}}|_h^2 \right\} \\
+ K \max_{0 < t' < T} \sum_{i=1}^{N-1} h_i^{2k+4} \|u_i\|_{H^{k+4}(I_i)} + \max_{0 < t' < T} \|e^t_i\|_{L^2(I)}^2 \\
(5.8) \\
+ \frac{1}{\varepsilon} \Delta t \sum_{j=1}^{n-1} \left\{ - (D_x^2 \xi^{j+\frac{1}{2}}, \xi^{j+\frac{1}{2}})_h + |\xi^{j+\frac{1}{2}}|_h^2 \\
- (D_x^2 \xi^{j-\frac{1}{2}}, \xi^{j-\frac{1}{2}})_h + |\xi^{j-\frac{1}{2}}|_h^2 \right\} \\
+ \Delta t \sum_{j=1}^{n-1} \|e^j_i\|_{H^{k+3}(I)}^2 \\
+ K \Delta t \sum_{j=1}^{n-1} \sum_{i=0}^{N-1} h_i^{2k+4} \|u_i\|_{H^{k+3}(I_i)}^2 + \|D_x^2 u_i\|_{H^{k+3}(I_i)}^2, \\
\}
\]

where \(e^j_i = O(\Delta t^2)\). From (5.6)–(5.8) and the discrete form of the Gronwall Lemma we derive in [6] the relation
Finally from Lemma 1.4, 2.2 and inequality (5.9), we conclude that
\[
\|\xi\|_{L^\infty_{\Delta t}(0,T;L^\infty)} \leq C \left[ \|\xi\|_{H^1(I)} + \|\partial_t \xi\|_{L^2(I)} \right] + Kh^{k+2} \left[ \|u\|_{L^2_{\Delta t}(0,T;H^{k+3}(I))} + \|D^2_t u\|_{L^2_{\Delta t}(0,T;H^{k+4}(I))} \right] + c(u) A t^2,
\]
(5.10)

where \( C \) and \( K \) are generic constants independent of \( u, h, \Delta t \) and \( c(u) \) independent of \( h, A t \). From the results of Section 2 we easily see that
\[
\|\eta\|_{L^\infty_{\Delta t}(0,T;L^\infty)} \leq C h^{k+2} \|u\|_{L^\infty_{\Delta t}(0,T;H^{k+2})}.
\]
(5.11)

Therefore, the inequalities (5.10) and (5.11) imply
\[
\|u - u_h\|_{L^\infty_{\Delta t}(0,T;L^\infty)} \leq c(u) (h^{k+2} + (\Delta t)^2),
\]
provided
\[
\|\xi\|_{H^1(I)} + \|\partial_t \xi\|_{L^2(I)} \leq ch^{k+2},
\]
where \( c(u) \) is independent of \( h \) and \( \Delta t \). This concludes the proof of Theorem 5.1.

It remains to discuss the choice of \( u^0_h \) and \( u^1_h \). We choose \( u^0_h \equiv T_h u(x, 0) \) and \( u^1_h \equiv T_h \tilde{u} \) where
\[
\tilde{u} \equiv u(x, 0) + \Delta t D_t u(x, 0) + \frac{(\Delta t)^2}{2} D^2_t u(x, 0) + \frac{(\Delta t)^3}{6} D^3_t u(x, 0);
\]
the derivatives \( D^2_t u \) and \( D^3_t u \) are evaluated using the differential equation.

6. The Superconvergence Phenomenon. Consider the linear hyperbolic problem
\[
(6.1) \quad p(x, t) D^2_t u - D_x^2 u = f(x, t), \quad (x, t) \in (0, 1) \times (0, T),
\]
subject to initial conditions
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(6.2) \[ u(x, 0) = \varphi_1(x), \quad D_t u(x, 0) = \varphi_2(x), \quad 0 \leq x \leq 1, \]

and boundary conditions

(6.3) \[ u(0, t) = 0, \quad u(1, t) = 0, \quad 0 < t \leq T. \]

Also, we assume for all \((x, t) \in [0, 1] \times [0, T]\),

(6.4) \[ 0 < m < p(x, t) < M, \quad 0 < m < q(x, t) < M. \]

Let \(u_h\) denote the collocation on lines approximation defined from (3.5) and (3.6) where \(p, q\) and \(f\) are independent of \(u\). Throughout we denote by \(L = p D^2_t - D^2_x, \|u\|_{l,i} \equiv \sup \{D^a_x D^b_t u(x, t)|x \in I, \alpha \leq i, \beta \leq i\}\) and \(x_{i-1/2} \equiv (i - \frac{1}{2})h_j\). By Peano's Kernel Theorem [9] we obtain

\[
L(u - T_k u)(\xi_{k+i}, t) = \sum_{i=1}^{k-2} \{D^{k+i+1}_x D^2_t u(x_{j-i}) \psi_i(t)i) - D^{k+i+3}_x u(x_{j-i})\psi''_i(t)i)\}h^{k+i+1}_j
\]

\[
- D^{k+3}_x u(x_{j-i})\psi''_2(t)i)h^{k+1}_j + O(h^{k+s}[\|u\|_{k+s+2,0} + \|u\|_{k+s+2,2}]),
\]

where

\[
\psi_i(x) = \frac{1}{(k + i + 1)!} A_i(x) R_{k+2}(x)
\]

with \(A_i\) a polynomial of degree \(i - 1\). In order to cancel the term of \(h^{k+1}_j\) accuracy we make a correction to \(T_k u\) defined locally by the following relations, \(\delta_0(\cdot, t) \in P_{k+2, \Delta x} \cap C^1\) with

\[
h^{k-1}_j D^2_x \delta_0(\xi_{k+i}, t) = D^{k+3}_x u(x_{j-i})\psi''_2(t)i), \quad i = 1, \ldots, k, \quad j = 0, \ldots, N - 1,
\]

\[
\delta_0(x_j, t) = D_x \delta(x_j, t) = 0, \quad j = 0, 1, \ldots, N.
\]

Now, in order to cancel the \(h^{k+i+1}_j\) order terms we define a new correction in the following way: first we introduce the function

\[
u(y) = \begin{cases} 
0, & y \leq 0, \\
3y^2 - 2y^3, & 0 \leq y \leq 1, \\
1, & 1 \leq y,
\end{cases}
\]

which obviously belongs to \(C^1\) and define for \(x \in I_j\)

\[
E_j(x, t) \equiv \lambda_{1,i} D^{k+i+1}_x D^2_t u(x_{j-i})u\left(\frac{x - x_j}{h_j}\right) - \lambda_{2,i} D^{k+i+3}_x u(x_{j-i})u\left(\frac{x - x_j}{h_j}\right),
\]

where \(\lambda_{1,i} \equiv - \psi'(t)\psi''(t), \lambda_{2,i} \equiv - \psi''(t)\psi'(t).\) Also, we define
\[ \delta_i(x, t) \equiv \sum_{j=0}^{N-1} h_j^{k+3-s} \{ E_j(x, t) - x E_j(1, t) \} \]
\[ = \sum_{j=0}^{N-1} \{ \lambda_{1,1} D_x^{k+1+t+1} D_t^2 u(x_{j-\gamma_2}) - \lambda_{2,1} D_x^{k+1+t+3} u(x_{j-\gamma_3}) \} \left( v \left( \frac{x - x_j}{h_j} \right) - x \right). \]

In [6] we show that the \( \lambda_{\alpha,1} \) for \( \alpha = 1, 2 \) are well defined and easily obtain
\[ L(u - \bar{u}) (\xi_{kj+i}, t) = O(h_j^{k+s} [\|u\|_{k+s+2,0} + \|u\|_{k+s+2,2}]), \]
where
\[ \bar{u} = T_h u + h_j^{k+s} \sum_{l=0}^{s-2} \delta_l. \]

**Theorem 6.1.** Let \( u \) denote the solution of the problem (6.1) to (6.4) such that \( u \in L^\infty(0, T; H^{k+s+4}) \), \( s \leq k \), and \( u_h \) is the collocation on lines approximation of \( u \) defined by (3.5), (3.6). Then the error of approximation at the nodes satisfies
\[ \max_i \| (u - u_h)(x_i, \cdot) \|_{L^\infty(0, T)} \leq C h_j^{k+s} [\|u\|_{k+s+2,0} + \|u\|_{k+s+2,2}] \]
\[ + C [\| D_t(u_h - \bar{u}) \|_{L^2(I)}(0) + \| u_h - \bar{u} \|_{H^1(I)}(0)], \]
where \( C \) is a constant independent of \( u \) and \( h \), and \( s \leq k \).

**Proof.** We define
\[ \rho(\xi_{kj+i}, t) \equiv L(u_h - \bar{u}) (\xi_{kj+i}, t), \]
where
\[ |\rho(\xi_{kj+i}, t)| \leq C h_j^{k+s} [\|u\|_{k+s+2,0} + \|u\|_{k+s+2,2}]; \]
and we form the relation
\[ (\rho, D_t(u_h - \bar{u}))_h = (D_t^2(u_h - \bar{u}), D_t(u_h - \bar{u}))_h - (D_x^2(u_h - \bar{u}), D_t(u_h - \bar{u}))_h. \]

We apply the elementary inequality \( \frac{1}{2} a^2 + \frac{1}{2} b^2 \geq ab \) to obtain
\[ |\rho|^2_h + |D_t(u_h - \bar{u})|^2_h \geq D_t |D_t(u_h - \bar{u})|^2_h - D_t (D_x^2(u_h - \bar{u}), u_h - \bar{u})_h. \]

In the above inequality we add the inequality
\[ \frac{1}{2} D_t |u_h - \bar{u}|^2_h \leq \frac{1}{2} |D_t(u_h - \bar{u})|^2_h + \frac{1}{2} |u_h - \bar{u}|^2_h \]
to obtain
\[ |\rho|^2_h + |u_h - \bar{u}|^2_h + 2|D_t(u_h - \bar{u})|^2_h \geq D_t \{ |u_h - \bar{u}|^2_h + |D_t(u_h - \bar{u})|^2_h \} - D_t (D_x^2(u_h - \bar{u}), u_h - \bar{u})_h. \]

We integrate from 0 to \( t \) and apply Gronwall's Lemma to get
\[ C \int_0^T |\rho_h^2(\tau)|d\tau + |u_h - \overline{u}|^2_h(0) + |D_t(u_h - \overline{u})|^2_h(0) - (D_x^2(u_h - \overline{u}), u_h - \overline{u})_h(0) \]

\[ \geq |u_h - \overline{u}|^2_h + |D_t(u_h - \overline{u})|^2_h - (D_x^2(u_h - \overline{u}), u_h - \overline{u})_h. \]

It follows from Lemmas 1.3, 1.4, 1.5 that

\[ C \left\{ \max_t |\rho_h|_h + \|D_t(u_h - \overline{u})\|_{L^2(I)}(0) + \|u_h - \overline{u}\|_{H^1(I)}(0) \right\} \]

\[ \geq \|u_h - \overline{u}\|_{H^1(I)} + |D_t(u_h - \overline{u})|^2_h. \]

In particular, we have

\[ C \left\{ \max_t |\rho_h|_h + \|D_t(u_h - \overline{u})\|_{L^2(I)}(0) + \|u_h - \overline{u}\|_{H^1(I)}(0) \right\} \geq \|u_h - \overline{u}\|_{L^\infty(I)}. \]

It is easy to see that

\[ |(u - \overline{u})(x_j, t)| \leq C h^{k+s}[\|u\|_{k+s-1,2} + \|u\|_{k+s+1,0}], \]

where \( h = \max_j h_j \). Consequently, we have

\[ \max_t \max_{0 \leq j \leq N} |(u - u_h)(x_j, t)| \leq C h^{k+s}[\|u\|_{k+s+2,0} + \|u\|_{k+s+2,2}] \]

\[ + C \{\|D_t(u_h - \overline{u})\|_{L^2(0)} + \|u_h - \overline{u}\|_{H^1(I)}(0)\}. \]

This completes the proof of Theorem 6.1.

Now, we consider the problem of choosing initial values, in order to obtain maximum accuracy. It is clear that the following

\[ u_h(x, 0) = T_h \phi_1 + h^{k+s} \delta_0(x, 0), \quad D_t u_h(x, 0) = T_h \phi_2 + h^{k+s} \delta_0(x, 0) \]

yields \( (u_h - \overline{u})(0) = O(h^{k+s}) \) in \( H^1 \) norm, and \( D_t (u_h - \overline{u})(0) = O(h^{k+s}) \) in \( L^2 \) norm.

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