Application of Method of Collocation on Lines for Solving Nonlinear Hyperbolic Problems

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Abstract. A collocation on lines procedure based on piecewise polynomials is applied to initial/boundary value problems for nonlinear hyperbolic partial differential equations. Optimal order a priori estimates are obtained for the error of approximation. The Crank-Nicholson discretization in time is studied and convergence rates of the collocation-Crank-Nicholson procedure are established. Finally, the superconvergence is verified at particular points for linear hyperbolic problems.

Introduction. We consider the nonlinear hyperbolic problem

\[ p(x, t, u) \frac{\partial^2 u}{\partial x^2} - q(x, t, u) \frac{\partial u}{\partial x} = f(x, t, u, \frac{\partial u}{\partial x}), \quad (x, t) \in (0, 1) \times (0, T], \]

subject to the initial conditions

\[ u(x, 0) = u_0, \quad \frac{\partial u}{\partial t}(x, 0) = u_1, \quad x \in (0, 1), \]

and to Dirichlet boundary conditions for \( t > 0 \). We examine the convergence of the collocation on lines procedure using piecewise polynomials with continuous first derivatives as the approximating functions.

In Section 4 we obtain optimal-order asymptotic estimates for the error of the approximation in the \( L_\infty \)-norm. In Section 5, the Crank-Nicholson discretization of the resulting system of ordinary differential equations is studied and convergence rates of the collocation on lines--Crank-Nicholson procedure are established. Finally, in Section 6 the superconvergence phenomenon is established locally for a linear hyperbolic problem.

The method of collocation on lines was proposed first by Kantorovich [7]. The convergence of this method for a problem of mathematical physics was investigated by E. B. Karpilovskaya [8]. Yartsev [11], [10] proved convergence for linear elliptic and biharmonic type problems using trigonometric polynomials as basis functions. Douglas and Dupont [3], have studied the same method using piecewise cubic Hermite polynomials for a nonlinear parabolic problem and in [4] verified the superconvergence locally for the heat equation. Finally, Douglas and Dupont [5] generalized and extended their results in [3], [4]. The results in this paper are from the author’s thesis [6].

1. Preliminary Results. Let \( \Delta_x = (x_i)_{i=0}^N \) be a partition of \([0, 1]\), \( I = [0, 1] \), \( h_i \equiv |x_{i+1} - x_i| \), \( I_j \equiv [x_j, x_{j+1}] \) and \( h \equiv \max_j |x_{j+1} - x_j| \). Throughout this paper

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we denote by $P_r$ the set of polynomials of degree less than $r$ and $P_{r, \Delta x}$ the set of functions that are polynomials of degree $r - 1$ in each subinterval $[x_i, x_{i+1}]$. We take $-1 < \rho_1 < \rho_2 < \cdots < \rho_k < 1$ and $w_j > 0, j = 1, \ldots, k,$ to be Gaussian points and weights, respectively, so that

\[ \int_{-1}^{+1} p(x) dx = \sum_{i=1}^{k} p(\rho_i)w_i, \quad p \in P_{2k}([-1, 1]) . \]

The Gaussian points and weights in the subinterval $[x_i, x_{i+1}]$ are

\[ \xi_{k,i} = (x_i + x_{i+1})/2 + \rho_i h_i/2, \quad w_i^* = h_i w_i/2, \quad i = 1, \ldots, k. \]

We introduce two pseudo-inner products corresponding to Gaussian quadrature and composite Gaussian quadrature:

\[ (f, g)_{h_j} = \frac{h_j}{2} \sum_{i=1}^{k} w_i f(\xi_{k,i}) \cdot g(\xi_{k,i}), \]

and

\[ (f, g)_h = \sum_{j=0}^{N-1} (f, g)_{h_j}, \]

with

\[ |f|_h = \sum_{j=0}^{N-1} (f, f)_{h_j}. \]

For later use, we state without proof the lemmas:

**Lemma 1.1.** The seminorm $|f|_h$ is positive definite for all $f \in P_{k+2, \Delta x} \cap C^1 [0, 1]$ with $f(0) = f(1) = 0$.

**Lemma 1.2.** If $f, g \in P_{k+2, \Delta x} \cap C^1 [0, 1]$, then

\[ -(D_x^2 f, g)_h = (D_x f, D_x g) - D_x f \cdot g|_0^1 \]

\[ + \frac{(k + 1)k}{(2k)!} \sum_{j=1}^{k} \frac{D_x^{k+1} f_j}{(k + 1)!} \frac{D_x^{k+1} g_j}{(k + 1)!} \int_{x_i}^{x_{i+1}} \prod_{l=1}^{k} (x - \xi_{k,l})^2 dx. \]

**Lemma 1.3.** If $f \in \{ v \in P_{k+2, \Delta x} \cap C^1, v(0) = v(1) = 0 \}$, then

\[ (D_x f, D_x f) \leq (D_x^2 f, f)_h \leq 2(D_x f, D_x f) \]

and

\[ |D_x f|^2_h \leq (D_x f, D_x f). \]

**Lemma 1.4.** If $f \in P_{k+2, \Delta x} \cap C^1 [0, 1]$, then

\[ |f|_h \leq \lambda \| f \|_{L^2(I)}, \]

where $\lambda$ is the maximum eigenvalue of the matrix $A_{k+1} \equiv \left[ \Sigma_{i=1}^{k} w_i L_i(\rho_i)L_j(\rho_j) \right]$ and $L_i$ denotes the $i$th degree Legendre polynomials in $[-1, 1]$.

Let $H^k$ be the Sobolev space of functions having $L^2$-derivatives of order $k$ on $I$ and $H_0^k \equiv \{ u \in H^k| u(0) = u(1) = 0 \}$. 

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Lemma 1.5. For $f \in H^1$ we have

$$\langle D_x f, D_x f \rangle + \|f\|_{H^1(I)}^2 \geq \frac{1}{4} \|f\|^2_{H^1(I)}.$$

The above lemmas are established in [6], proofs also appear in [5]. Lemmas 1.2, 1.3 and 1.5 have been first proved for the case of cubic Hermite polynomials by Douglas and Dupont [3].

2. Approximation Theory. In [6] we show that $R_k(x) \equiv D_x^k(1 - x^2)^{k+2}$, $k = 0, 1, \ldots$, on $(-1, 1)$ are orthogonal polynomials. By Rodrigues’ formula we see that $D_x^2 R_k(x) = D_x^{k+2}(1 - x^2)^{k+2}$ is a multiple of the Legendre polynomial on the interval $(-1, 1)$. We now establish some properties of these polynomials.

Lemma 2.1. If $k \geq 3$,

$$\langle D_x R_{k-2}, x^\mu \rangle_h = 0, \quad \mu = 0, 1, 2, \nu \leq \mu.$$

Proof. Since $D_x^\mu R_{k-2} x^\mu$ is a polynomial of degree $K + 2 - \mu + \nu$, we have for $k \geq 3$,

$$\langle D_x^\mu R_{k-2}, x^\nu \rangle_h = \int_{-1}^{1} D_x^\nu R_{k-2} x^\nu dx.$$

Lemma 2.1 now follows by using integration by parts and the fact that $D_x^\mu R_{k-2}$ vanishes at $x = \pm 1$ and $D_x^2 R_{k-2}$ vanishes at the Gaussian points. Note that for $k \geq 2$,

$$\langle D_x R_{k-2}, 1 \rangle_h = \langle D_x^2 R_{k-2}, x^\nu \rangle_h = 0.$$

We define an interpolation operator

$$T_h : C^1(I) \rightarrow P_{k+2, \Delta_x} \cap C^1(I)$$

such that

$$\langle T_h v \rangle(x_i) = v(x_i),$$

$$\langle D_x T_h v \rangle(x_i) = (D_x v)(x_i), \quad l = 0, 1, \ldots, N,$$

$$\langle T_h v \rangle (\tau_{(i,j)}) = (v(\tau_{(i,j)}), \quad i = 1, \ldots, k, j = 1, \ldots, N,$$

where $\tau_{(i,j)} = x_j + a_i (x_{j+1} - x_j)$ and the $a_i$'s are the roots in the interval $(0, 1)$ of the orthogonal polynomials $R_{k-2}(x)$.

Lemma 2.2. Assume that $u \in H^{k+4}(I)$ and let $e \equiv u - T_h u$. Then there is a constant $K$ independent of $h$ so that

$$|D_l^l e|_h \leq Kh^{k-l+2} \|u\|_{H^{k+2}(I)}, \quad l = 0, 1,$$

$$|D_x^2 e|_h \leq Kh^{k-1} \|u\|_{H^{k+3}(I)},$$

$$\langle D_x e, 1 \rangle_h \leq Kh^{k+5/2} \|u\|_{H^{k+3}(I)},$$

$$\langle D_x^2 e, 1 \rangle_h \leq Kh^{k+5/2} \|u\|_{H^{k+4}(I)}.$$

Proof. It follows easily from Lemma 2.1 and Peano’s Kernel Theorem [9].
3. Collocation on Lines. In this section we consider the problem of approximating the solution of the nonlinear hyperbolic equation

\[(3.1)\quad p(x, t, u)D^2_x u - q(x, t, u)D^2_x u = f(x, t, u, Dx u), \quad (x, t) \in (0, 1) \times (0, T],\]

subject to the initial conditions

\[(3.2)\quad u(x, 0) = \alpha_1(x), \quad D_t u(x, 0) = \alpha_2(x), \quad 0 < x < 1,
\]

and the boundary conditions

\[(3.3)\quad u(0, t) = 0, \quad u(1, t) = 0, \quad 0 < t \leq T.
\]

Assume that the coefficients satisfy

\[(3.4)\quad 0 < c_1 \leq p(x, t, u) \leq C_1, \quad c_2 \leq q(x, t, u) \leq C_2,
\]

for \(0 \leq x \leq 1, 0 \leq t \leq T\) and \(-\infty < u < +\infty\). Also, we assume that \(p, q, f\) are continuously differentiable functions of their arguments and uniformly bounded.

Throughout, we assume that this problem has a solution, \(u\).

Let \(\Delta_x^k \equiv P_{k+2, \Delta_x} \cap C^1\) and \(\Delta_x^k \cap H^1_0\) be spanned by the basis functions \(\{B_j\}_{kN}\). We seek an approximation \(u_h(x, t)\) to \(u\) of the form

\[u_h(x, t) = \sum_{i=1}^{kN} \beta_i(t)B_i(x).
\]

The coefficients \(\{\beta_i(t)\}_{i=1}^{kN}\) as functions of time are the solutions of the nonlinear ordinary differential equations

\[(3.5)\quad (p(u_h)D^2_t u_h - q(u_h)D^2_x u_h - f(u_h, D_x u_h))(\xi_i, t) = 0, \quad 0 < t \leq T, i = 1, \ldots, kN,
\]

and

\[(3.6)\quad u_h(\xi_i, 0) = \hat{\alpha}_1(\xi_i), \quad D_t u_h(\xi_i, 0) = \hat{\alpha}_2(\xi_i), \quad k = 1, \ldots, kN,
\]

where \(\hat{\alpha}_1, \hat{\alpha}_2\) are the \(\Delta_x^k\)-interpolants of \(\alpha_1(x), \alpha_2(x)\) respectively.

Although these are the equations which one solves in practice, the analysis is more conveniently made if one considers the equivalent problem of finding \(u_h \in \Delta_x^k \cap H^1_0\) such that

\[(3.7)\quad (p(u_h)D^2_t u_h - q(u_h)D^2_x u_h - f(u_h, D_x u_h), B_i)_h = 0, \quad 0 < t \leq T, i = 1, \ldots, kN,
\]

and

\[(3.8)\quad u_h(\xi_i, 0) = \hat{\alpha}_1(\xi_i), \quad D_t u_h(\xi_i, 0) = \hat{\alpha}_2(\xi_i), \quad i = 1, \ldots, kN.
\]

**Lemma 3.1.** The collocation method (3.5), (3.6) and the discrete Galerkin method (3.7), (3.8) each possess a unique solution for \(0 < t \leq T\). Moreover, these solutions are identical if the processes are started from the same initial values.

**Proof.** It follows from Lemma 4.1 in [5].
4. Error Analysis. In this section, we find a priori error bounds for the collocation on lines procedure. We consider the problem of finding $u_h \in S_{\Delta x} \cap H^1_0$ such that

$$\tag{4.1} (p(u_h) D^2_t u_h - D^2_x u_h - f(u_h, D_x u_h), v)_h = 0, \quad 0 < t \leq T,$$

for all $v \in S_{\Delta x} \cap H^1_0$.

In order to find estimates for the error $u - u_h$ in the $L_\infty$-norm, we assume that $u(\cdot, t) \in C^1(I)$ and define $w(\cdot, t) \equiv T_h u$ which is in $S_{\Delta x}$. Then we find a priori bounds for the difference $w - u_h \in S_{\Delta x}$; and applying known approximation results to the difference $u - w$, we obtain bounds for the error of the collocation on lines procedure.

If $X$ is a normed space and $\psi : [0, T] \rightarrow X$, define

$$\| \psi \|_{L^2(0, T; X)} = \int_0^T \| \psi(t) \|_X^2 \, dt, \quad \| \psi \|_{L_\infty(0, T; X)} = \sup_{0 < t \leq T} \| \psi(t) \|_X.$$ 

**Theorem 4.1.** If

(i) the coefficients in (3.1) have bounded third derivatives and satisfy conditions (3.4),

(ii) $u \in L^\infty(0, T; H^{k+4})$, $D_t u \in L^2(0, T; H^{k+4})$ and $D^2_t u \in L^2(0, T; H^{k+4})$, 

(iii) $u_h(x, 0)$, $D_t u_h(x, 0)$ are the $S_{\Delta x}$ interpolants of $u(x, 0)$ and $D_t u(x, 0)$, respectively, then for the error of approximation we have

$$
\| u - u_h \|_{L_\infty(0, T; L^2)} \leq K \left[ \| u \|_{L^\infty(0, T; H^{k+4}(I))} + \| D_t u \|_{L^2(0, T; H^{k+4}(I))} 
+ \| D^2_t u \|_{L^2(0, T; H^{k+4}(I))} \right] h^{k+2},
$$

where $K$ is a constant independent of $h$ and $u$.

**Proof.** Let $\eta \equiv u - w$ and $\xi \equiv w - u_h$. Then (3.1), (4.1) imply that

$$
(p(u_h) D^2_t \xi - D^2_x \xi, v)_h = (-p^1 \eta D^2_t w - p(w) D^2_t \eta - p^2 \eta D^2_x u, v)_h
+ (D^2_x \eta, v)_h + \left( [f(w, D_x u) - f(w, D_x w)], v \right)_h
+ (f^1 \eta + f^2 \xi + f^3 D_x \xi, v)_h.
$$

In (4.2) we choose $v = D_t \xi$ and in [6] we show that

$$
\frac{1}{2} \left[ \sqrt{p(u_h)} D^2_t \xi_h^2 + |\xi|^2_h - (D^2_x \xi, \xi)_h \right]
\leq K \int_0^f \left( |\eta|^2_h + |D_x \xi|^2_h \right) \, dt + \int_0^f \left( |\eta|^2_h + |D^2_t \eta|^2_h \right) \, dt
+ K \int_0^f |D_t \xi|^2_h
\leq K \left\{ (\xi_h^2(0) - (D^2_x \xi, \xi)_h(0)) + \sqrt{p(u_h)} D_t \xi_h^2(0) \right\}
+ \int_0^f (D^2_x \eta, D_t \xi)_h \, dt + \int_0^f (f(w, D_x u) - f(w, D_x w), D_t \xi)_h \, dt.
$$

Integration by parts gives

$$
\int_0^f (D^2_x \eta, D_t \xi)_h \, dt = (D^2_x \eta, \xi)_h|_0^f - \int_0^f (D_t D^2_x \eta, \xi)_h \, dt,
$$
and

\[ \int_0^t (f(w, D_x u) - f(w, D_x w), D_t \xi)_h \, d\tau \]

\[ = (f(w, D_x u) - f(w, D_x w), \xi)_h \big|_0^t - \int_0^t (D_t \{ f(w, D_x u) - f(w, D_x w) \}, \xi)_h \, d\tau. \]

Using Poincaré's inequality, the elementary inequality \(|cd| \leq (\lambda p)c^2 + pd^2\) and Lemma 2.2 in [6] we have obtained

\[ \left| \int_0^t (D_t D_x^2 \eta, \xi)_h \, d\tau \right| \leq \frac{1}{16} \int_0^t \left[ -(D_x^2 \xi, \xi)_h + |\xi|^2_h \right] \, d\tau \]

(4.4)

\[ + K \sum_{i=1}^N h_j^{2k+4} \int_0^t \| D_t u(\cdot, \tau) \|_{H^{k+4}(I_j)}^2 \, d\tau. \]

Using Taylor's theorem, we can easily show that

\[ (f(w, D_x u) - f(w, D_x w), w - u_h)_h \]

\[ = \sum_{j=1}^N (f_{D_x} w(w, D_x w)|_{x=t_{k(j-1)+1}}^1 D_x \eta + \omega h_j D_x \eta, \xi)_h, \]

where \(\omega\) is bounded independent of \(h_j\). It follows from Lemma 2.2

\[ \|f_{D_x} w(w, D_x w)|_{x=t_{k(j-1)+1}}^1 D_x \eta + \omega h_j D_x \eta, \xi\|_h \]

\[ \leq Kh_j^{k+2} \left[ \|u(\cdot, \tau)\|_{H^{k+3}(I_j)} \|\xi\|_h + \|u(\cdot, \tau)\|_{H^{k+3}(I_j)} \|D_x \xi(\cdot, \tau)\|_{L^2(I_j)} \right], \]

and

\[ \|\omega h_j D_x \eta, \xi\|_h \leq Kh_j^{k+2} \|u(\cdot, \tau)\|_{H^{k+2}(I_j)} \|\xi\|_h. \]

Moreover, we obtain

\[ \|f(w, D_x u) - f(w, D_x w), \xi\|_h \]

\[ \leq \frac{1}{16} \left[ -(D_x^2 \xi, \xi)_h + |\xi|^2_h \right] + K \sum_{j=1}^N h_j^{2k+4} \|u(\cdot, \tau)\|_{H^{k+3}(I_j)}^2. \]

Following similar arguments as above, we show that

\[ \int_0^t (D_t \{ f(w, D_x u) - f(w, D_x w) \}, \xi)_h \, d\tau \]

(4.5)

\[ \leq \frac{1}{16} \int_0^t \left[ -(D_x^2 \xi, \xi)_h + |\xi|^2_h \right] \, d\tau \]

\[ + K \sum_{j=1}^N h_j^{2k+4} \int_0^t \|u(\cdot, \tau)\|_{H^{k+3}(I_j)}^2 + \|D_t u(\cdot, \tau)\|_{H^{k+3}(I_j)}^2 \, d\tau. \]

It follows from (4.3)-(4.5), (1.3) and Gronwall's Lemma [6] that
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\[ \| w - u_h \|_{L^\infty(0,T;L^\infty)}^2 \leq K \left[ \| (w - u_h) (\cdot, 0) \|_{H^1(I)}^2 + \| D_t (w - u_h) (\cdot, 0) \|_{L^2(I)}^2 \right. \]

\[ + \sum_{j=1}^N h_j^{2k+4} \| u \|_{L^\infty(0,T;H^{k+4}(I_j))}^2 + \| D_t u \|_{L^2(0,T;H^{k+4}(I_j))}^2 + \| D^2_t u \|_{L^2(0,T;H^{k+4}(I_j))}^2 \}. \]

It is an elementary consequence of Peano's Kernel Theorem that

\[ \| u - w \|_{L^\infty(0,T;L^\infty)}^2 \leq K \sum_{j=1}^N h_j^{2k+4} \| u \|_{L^\infty(0,T;H^{k+4}(I_j))}^2. \]

Finally, from (4.6), (4.7) and assumption (iii) it follows that

\[ \| u - u_h \|_{L^\infty(L^\infty(I))} \]

\[ \leq Kh^{k+2} \| u \|_{L^\infty(H^{k+4}(I_j))} + \| D_t u \|_{L^2(H^{k+4}(I_j))} + \| D^2_t u \|_{L^2(H^{k+4}(I_j))}. \]

This concludes the proof of Theorem 4.1.

5. Computational Considerations. In this section, we discuss the question of actually solving the system of ordinary differential equations (3.5), (3.6).

Let

\[ u_h^0 \equiv u_h^0(x) = u_h^0(x', t'), \quad t' = t + \Delta t, \quad \Delta t = T/N, \]

\[ v^{j+\frac{1}{2}} = (v^{j+1} + v^j)/2, \quad v^{j,\frac{1}{2}} = \frac{1}{4} v^{j+1} + \frac{1}{4} v^j + \frac{1}{4} v^{j-1}, \]

\[ \partial_t v^{j+\frac{1}{2}} = (v^{j+1} - v^j)/\Delta t, \quad \partial_t^2 v^j = (v^{j+1} - 2v^j + v^{j-1})/(\Delta t)^2. \]

Then the Crank-Nicholson-Collocation approximation \{u_h^i\}_0^N is defined such that

\[ (p(t', u_h^{i,\frac{1}{2}}) \partial_t^2 u_h^i - q(t', u_h^{i,\frac{1}{2}}) D_x^2 u_h^{i,\frac{1}{2}} - f(t', u_h^{i,\frac{1}{2}}, D_x u_h^{i,\frac{1}{2}}))(x_i) = 0, \]

\[ i = 1, \ldots, kN, j = 0, \ldots, N - 1, \]

\[ (ii) \quad u_h^0(0) = u_h^0(1) = 0, \quad j = 0, \ldots, N. \]

At the end of this section we discuss the choice of \( u_h^0, u_h^1 \). In order to analyze the convergence of the solution of (5.2) we consider the equivalent to (5.2) normalized problem

\[ (p(t', u_h^{i,\frac{1}{2}}) \partial_t^2 u_h^i, v)_h - (D_x^2 u_h^{i,\frac{1}{2}}, v)_h = (f(t', u_h^{i,\frac{1}{2}}, D_x u_h^{i,\frac{1}{2}}), v)_h, \]

\[ v \in S_{\Delta x} \cap H^1_0, 0 < j < N. \]

Also, we introduce the notation

\[ \| u \|_{L^2_{\Delta t}(0,T;X)}^2 = \sum_{0 < t < T} \| u \|_{X_t}^2 \Delta t, \]

\[ \| u \|_{L^\infty_{\Delta t}(0,T;X)}^2 = \max_{0 < t < T} \| u \|_{X_t}^2, \]

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\[ \|u\|_{L^2_{\Delta t}(0,T;X)}^2 \equiv \sum_{0 \leq i \leq T} \|u_i\|_{X}^2 \Delta t. \]

**Theorem 5.1.** Assume the hypotheses (i), (ii) of Theorem 4.1 hold. Further, assume \( D^3_x u, D^4_x u \) are in \( L^\infty(0, T; L^2(I)) \) and

\[ \|(u_h - w)\|_{H^1(I)} + \|\partial_x(u_h - w)\|_{L^2(I)} = O(h^{k+2}). \]

For \( \Delta t \) sufficiently small there exists a unique solution of the Crank-Nicholson-Collocation equations (5.2) and for the error of approximation we have

\[ \|u - u_h\|_{L^\infty_{\Delta t}(0,T;L^\infty)} \leq C(h^{k+2} + (\Delta t)^2), \]

where \( C \) depends on \( u \) and is independent of \( h, \Delta t \).

**Proof.** It is easily seen that a unique solution of (5.2) exists under assumption (i) and (3.3) for \( \Delta t \) sufficiently small. Throughout this proof we use the notation \( w \equiv T_h u, \eta \equiv u - w \) and \( \xi \equiv u_h - w \). First, we observe that \( u \) satisfies

\[ (p(u^{j+1/2}, \partial_t^2 \xi), v)_h = (p(u^{j+1/2}, D_x^2 u^{j+1/2}, v)_h + (\varepsilon, v)_h \]

for \( v \in S_{\Delta x} \cap H^1(I) \), where \( \|v\|_{L^2(I)} = O(\Delta t^2)\|D^4_x u\|_{L^2(I)} \).

After straightforward calculations and the application of the Mean Value Theorem, we obtain

\[ (p(u^{j+1/2}, \partial_t^2 \xi, v)_h = (p(u^{j+1/2}, \partial_t^2 \xi, v)_h + (\varepsilon, v)_h \]

(5.5)

\[ + (f(u^{j+1/2}, D_x^2 u^{j+1/2}) v)_h \]

In (5.5), we choose as test function \( v = (\xi^{j+1} - \xi^{j-1})/2t \) and then we obtain

\[ \frac{1}{2\Delta t} \left[ (\sqrt{p(u^{j+1/2})}) \eta^{j+1/2} \right]_h^2 + |\xi^{j+1/2}|^2 - (D_x^2 \xi^{j+1/2}, \xi^{j+1/2})_h \]

\[ + \left[ \left( D_x^2 \eta^{j+1/4}, \frac{\xi^{j+1} - \xi^{j-1}}{2\Delta t} \right)_h \right] \]

\[ + \left[ \left( F(w^{j+1/4}, D_x w^{j+1/4}) - F(w^{j+1/4}, D_x u^{j+1/4}, \frac{\xi^{j+1} - \xi^{j-1}}{2\Delta t}) \right)_h \right], \]

where \( C \) is a generic constant.
Following the same arguments as in Section 4 and using Lemma 2.2, we get

\[
\Delta t \sum_{j=1}^{n-1} \left( D_x^2 \eta^{j,k} \frac{x^{j+1} - x^{j-1}}{2\Delta t} \right)_h
\]

\[
\leq \frac{1}{\varepsilon} \left\{ - (D_x^2 x^{n-\frac{1}{2}}, x^{n-\frac{1}{2}})_h + |x^{n-\frac{1}{2}}|_h^2 - (D_x^2 x^{\frac{1}{2}}, x^{\frac{1}{2}})_h + |x^{\frac{1}{2}}|_h^2 \right\}
\]

\[
+ K \max_{0 < t' < T} \sum_{i=1}^{N-1} h_i^{2k+4} \|u^{j+1}_i\|_{H^{k+4}(I_i)} + \max_{0 < t' < T} \|e_{j_i}^{2}\|_{L^2(I)}
\]

\[
(5.7)
\]

\[
+ \frac{1}{2\varepsilon} \Delta t \sum_{j=1}^{n-1} \left\{ - (D_x^2 x^{j+\frac{1}{2}}, x^{j+\frac{1}{2}})_h - (D_x^2 x^{j-\frac{1}{2}}, x^{j-\frac{1}{2}})_h \right\}
\]

\[
+ K \Delta t \sum_{j=1}^{n-1} \sum_{i=0}^{N-1} h_i^{2k+4} \|D_t^2 u^j\|_{H^{k+4}(I_i)}^2
\]

\[
+ \Delta t \sum_{j=1}^{N-1} \|e_{j_i}^{2}\|_{L^2(I)}^2,
\]

where \(\|e_{j_i}^{2}\|_{L^2(I)}^2 = O(\Delta t^2)\) for \(s = 1, 2\), \(K\) is a generic constant and \(\varepsilon\) a constant that can be small enough.

Finally, by arguments similar to those of Section 4 we can show that

\[
\Delta t \sum_{j=1}^{n-1} \left( f(w^{j,k}, D_x u^{j,k}) - f(w^{j,k}, D_x w^{j,k}), \frac{x^{j+\frac{1}{2}} - x^{j-\frac{1}{2}}}{\Delta t} \right)_h
\]

\[
\leq \frac{1}{\varepsilon} \left\{ - (D_x^2 x^{n-\frac{1}{2}}, x^{n-\frac{1}{2}})_h + |x^{n-\frac{1}{2}}|_h^2 - (D_x^2 x^{\frac{1}{2}}, x^{\frac{1}{2}})_h + |x^{\frac{1}{2}}|_h^2 \right\}
\]

\[
+ K \max_{0 < t' < T} \sum_{i=1}^{N-1} h_i^{2k+4} \|u^{j+1}_i\|_{H^{k+4}(I_i)} + \max_{0 < t' < T} \|e_{j_i}^{2}\|_{L^2(I)}
\]

\[
+ \frac{1}{\varepsilon} \Delta t \sum_{j=1}^{n-1} \left\{ - (D_x^2 x^{j+\frac{1}{2}}, x^{j+\frac{1}{2}})_h - (D_x^2 x^{j-\frac{1}{2}}, x^{j-\frac{1}{2}})_h \right\}
\]

\[
+ K \Delta t \sum_{j=1}^{n-1} \sum_{i=0}^{N-1} h_i^{2k+4} \|D_t^2 u^j\|_{H^{k+4}(I_i)}^2 + \|D_t^2 u^j\|_{H^{k+4}(I_i)}^2,
\]

\[
(5.8)
\]

where \(e_{j_i}^{2} = O(\Delta t^2)\). From (5.6)–(5.8) and the discrete form of the Gronwall Lemma we derive in [6] the relation
\[
\left| \partial_t \xi^{n-\frac{1}{2}} \right|^2 + \left| \xi^{n-\frac{1}{2}} \right|^2 - (D_x^2 \xi^{n-\frac{1}{2}}, \xi^{n-\frac{1}{2}})_h \\
\leq C \left\{ - (D_x^2 \xi^{\frac{1}{2}}, \xi^{\frac{1}{2}})_h + \left| \xi^{\frac{1}{2}} \right|^2 \right\} \\
+ K\Delta t \sum_{j=1}^{n-1} \sum_{i=0}^{N-1} h_i^{2k+4} \left[ \|u|^2\right]_{H^{k+4}(I)} + \|D_t^2 u|^2\right]_{H^{k+4}(I)} \\
+ C\Delta t \sum_{j=1}^{n-1} \left[ \|\partial_t^2 \eta\|^2 \|\eta^{-\frac{1}{2}}\|^2 \right] + \Delta t \sum_{j=1}^{n-1} \left[ \|e_j^2\|^2 \|L_2(I)\| + \|e_j^2\|^2 \|L_2(I)\| \right] \\
+ K \max_{0 < t' < T} \sum_{i=0}^{N-1} h_i^{2k+4} \|u|^2\right]_{H^{k+4}(I)} + \max_{0 < t' < T} \|e_j^2\|^2 \|L_2(I)\| \right] \\
(5.9)
\]

Finally from Lemma 1.4, 2.2 and inequality (5.9), we conclude that

\[
\|\xi\|^2 \leq C \left[ \|\xi^{H_1(I)} \|_{H^1(I)} + \|\partial_t \xi^{H_2(I)} \|_{H^2(I)} \right] \\
+ K h^{k+2} \left[ \|u\|^2 \|L_2^2(0, T; H^{k+4}(I)) \| + \|D_t^2 u\|^2 \|L_2(0, T; H^{k+4}(I)) \| \right] \\
+ c(u) \Delta t^2,
\]

where \(C\) and \(K\) are generic constants independent of \(u\), \(h\), \(\Delta t\) and \(c(u)\) independent of \(h\), \(\Delta t\). From the results of Section 2 we easily see that

\[
\|\eta\|_{L_\infty^2(0, T; L^\infty)} \leq Ch^{k+2} \|u\|_{L_\infty^2(0, T; H^{k+4}(I))},
\]

Therefore, the inequalities (5.10) and (5.11) imply

\[
\|u - u_h\|^2 \|L_\infty(0, T; L^\infty) \| \leq c(u) (h^{k+2} + (\Delta t)^2),
\]

provided

\[
\|\xi^{H_1(I)}\| + \|\partial_t \xi^{H_2(I)}\| \leq c h^{k+2},
\]

where \(c(u)\) is independent of \(h\) and \(\Delta t\). This concludes the proof of Theorem 5.1.

It remains to discuss the choice of \(u_0^h\) and \(u_1^h\). We choose \(u_0^h \equiv T_h u(x, 0)\) and \(u_1^h \equiv T_h \tilde{u}\) where

\[
\tilde{u} \equiv u(x, 0) + \Delta t D_t u(x, 0) + \frac{(\Delta t)^2}{2} D_t^2 u(x, 0) + \frac{(\Delta t)^3}{6} D_t^3 u(x, 0);
\]

the derivatives \(D_t^2 u\) and \(D_t^3 u\) are evaluated using the differential equation.

6. The Superconvergence Phenomenon. Consider the linear hyperbolic problem

\[
p(x, t) D_t^2 u - D_x^2 u = f(x, t), \quad (x, t) \in (0, 1) \times (0, T),
\]

subject to initial conditions.
(6.2) \[ u(x, 0) = \varphi_1(x), \quad D_t u(x, 0) = \varphi_2(x), \quad 0 \leq x \leq 1, \]
and boundary conditions

(6.3) \[ u(0, t) = 0, \quad u(1, t) = 0, \quad 0 < t \leq T. \]

Also, we assume for all \((x, t) \in [0, 1] \times [0, T]\),

(6.4) \[ 0 < m < p(x, t) \leq M, \quad 0 < m < q(x, t) \leq M. \]

Let \(u_h\) denote the collocation on lines approximation defined from (3.5) and (3.6) where \(p, q\) and \(f\) are independent of \(u\). Throughout we denote by \(L \equiv pD_t^2 - D_x^2, \|u\|_{l, i} \equiv \sup \left\{ D_x^\alpha D_t^\beta u(x, t) | x \in I, \alpha \leq j, \beta \leq i \right\}\) and \(x_{i-1/2} \equiv (i - \frac{1}{2})h_j\). By Peano’s Kernel Theorem [9] we obtain

\[
L(u - T_h u) (x_{k+1}, t) = \sum_{i=1}^{k+1} \left( D_x^{k+1+i} u(x_{j-x_i}) \psi_i(\rho_i) - D_x^{k+i+1} u(x_{j-x_i}) \psi_{i+1}(\rho_i) \right) h_i^{k+1+i} - D_x^{k+1} u(x_{j-x_i}) \psi_2''(\rho_i) h_i^{k+1} + O(h_i^{k+i+1})
\]

where

\[
\psi_i(x) = \frac{1}{(k + i)!} A_i(x) R_{k-i}(x)
\]

with \(A_i\) a polynomial of degree \(i - 1\). In order to cancel the term of \(h_i^{k+1}\) accuracy we make a correction to \(T_h u\) defined locally by the following relations, \(\delta_0(\cdot, t) \in P_{k+2}, \Delta_x \subset C^1\) with

\[
h_x^{i-1} D_x^2 \delta_0(x_{k+1}, t) = D_x^{k+3} u(x_{j-x_i}) \psi_2''(\rho_i), \quad i = 1, \ldots, k, \, j = 0, \ldots, N - 1,
\]

\[
\delta_0(x_j, t) = D_x \delta(x_j, t) = 0, \quad j = 0, 1, \ldots, N.
\]

Now, in order to cancel the \(h_i^{k+i+1}\) order terms we define a new correction in the following way: first we introduce the function

\[
u(y) = \begin{cases} 0, & y \leq 0, \\ 3y^2 - 2y^3, & 0 \leq y \leq 1, \\ 1, & 1 \leq y, \end{cases}
\]

which obviously belongs to \(C^1\) and define for \(x \in I_j\)

\[
E_j(x, t) = \lambda_{1,i} D_x^{k+i+1} D_t^2 u(x_{j-x_i}) v \left( \frac{x - x_i}{h_j} \right) - \lambda_{2,i} D_x^{k+i+3} u(x_{j-x_i}) v \left( \frac{x - x_i}{h_j} \right),
\]

where \(\lambda_{1,i} \equiv -\psi_i(\rho_i)/\psi_2''(\rho_i), \lambda_{2,i} \equiv -\psi_{i+2}(\rho_i)/\psi_2''(\rho_i)\). Also, we define
\[
\delta_t(x, t) \equiv \sum_{j=0}^{N-1} h_j^{l+3-s} \{ E_j(x, t) - xE_j(1, t) \}
\]
\[
= \sum_{j=0}^{N-1} \left\{ \lambda_1, l D_x^{k+l+1} \partial_t^2 u(x_{j+1}) - \lambda_2, l D_x^{k+l+3} u(x_{j-1}) \right\} \left( \frac{x - x_j}{h_j} \right) - x.
\]

In [6] we show that the \( \lambda_{\alpha, l} \) for \( \alpha = 1, 2 \) are well defined and easily obtain

\[
L(u - \bar{u}) (\xi_{kj+i}, t) = O(h_j^{k+s} [\|u\|_{k+s+2,0} + \|u\|_{k+s+2,2}]),
\]
where

\[
\bar{u} = T_h u + h_j^{k+s} \sum_{l=0}^{s-2} \delta_l.
\]

**Theorem 6.1.** Let \( u \) denote the solution of the problem (6.1) to (6.4) such that \( u \in L^\infty(0, T; H^{k+s+4}) \), \( s \leq k \) and \( u_h \) is the collocation on lines approximation of \( u \) defined by (3.5), (3.6). Then the error of approximation at the nodes satisfies

\[
\max_j \| (u - u_h)(x_j, \cdot) \|_{L^\infty(0, T)} \leq C h_j^{k+s} [\|u\|_{k+s+2,0} + \|u\|_{k+s+2,2}]
\]
\[
+ C [\| \partial_t (u_h - \bar{u}) \|_{L^2(I)}(0) + \| u_h - \bar{u} \|_{H^1(I)}(0)],
\]

where \( C \) is a constant independent of \( u \) and \( h \) and \( s \leq k \).

**Proof.** We define

\[
\rho(\xi_{kj+i}, t) \equiv L(u_h - \bar{u}) (\xi_{kj+i}, t),
\]
where

\[
| \rho(\xi_{kj+i}, t) | \leq C h_j^{k+s} [\|u\|_{k+s+2,0} + \|u\|_{k+s+2,2}];
\]

and we form the relation

\[
(\rho, \partial_t (u_h - \bar{u}))_h = (D_t^2 (u_h - \bar{u}), \partial_t (u_h - \bar{u}))(0) - (D_t^2 (u_h - \bar{u}), \partial_t (u_h - \bar{u}))(0).
\]

We apply the elementary inequality \( \frac{1}{2} a^2 + \frac{1}{2} b^2 \geq ab \) to obtain

\[
| \rho|^2 + | \partial_t (u_h - \bar{u}) |^2 \geq D_t | \partial_t (u_h - \bar{u}) |^2 - D_t (D_t^2 (u_h - \bar{u}), u_h - \bar{u})_h.
\]

In the above inequality we add the inequality

\[
\frac{1}{2} D_t | u_h - \bar{u} |^2_h \leq \frac{1}{2} | \partial_t (u_h - \bar{u}) |^2_h + \frac{1}{2} | u_h - \bar{u} |^2_h
\]

to obtain

\[
| \rho|^2 + | u_h - \bar{u} |^2_h + 2 | \partial_t (u_h - \bar{u}) |^2_h
\]
\[
\geq D_t \{ | u_h - \bar{u} |^2_h + | \partial_t (u_h - \bar{u}) |^2_h \} - D_t (D_t^2 (u_h - \bar{u}), u_h - \bar{u})_h.
\]

We integrate from 0 to \( t \) and apply Gronwall's Lemma to get
\[ C \int_0^T \left| \rho_h(t) \right|^2 dt + |u_h - \bar{u}|^2(0) + |D_t(u_h - \bar{u})|^2(0) - (D_x^2(u_h - \bar{u}), u_h - \bar{u})_h(0) \]

\[ \geq |u_h - \bar{u}|^2 + |D_t(u_h - \bar{u})|^2 - (D_x^2(u_h - \bar{u}), u_h - \bar{u})_h. \]

It follows from Lemmas 1.3, 1.4, 1.5 that

\[ C \left\{ \max_t |\rho|_h + \|D_t(u_h - \bar{u})\|_{L^2(I)}(0) + \|u_h - \bar{u}\|_{H^1(I)}(0) \right\} \]

\[ \geq \|u_h - \bar{u}\|_{H^1(I)} + |D_t(u_h - \bar{u})|^2. \]

In particular, we have

\[ C \left\{ \max_t |\rho|_h + \|D_t(u_h - \bar{u})\|_{L^2(I)}(0) + \|u_h - \bar{u}\|_{H^1(I)}(0) \right\} \geq \|u_h - \bar{u}\|_{L^\infty(I)}. \]

It is easy to see that

\[ |(u - \bar{u})(x, t)| \leq Ch^{k+s}[\|u\|_{k+s-1,2} + \|u\|_{k+s+1,0}], \]

where \( h = \max_j h_j \). Consequently, we have

\[ \max_{t} \max_{0 \leq j \leq N} |(u - u_h)(x_j, t)| \leq Ch^{k+s}[\|u\|_{k+s+2,0} + \|u\|_{k+s+2,2}] \]

\[ + C \{ \|D_t(u_h - \bar{u})\|_{L^2(I)}(0) + \|u_h - \bar{u}\|_{H^1(I)}(0) \}. \]

This completes the proof of Theorem 6.1.

Now, we consider the problem of choosing initial values, in order to obtain maximum accuracy. It is clear that the following

\[ u_h(x, 0) = T_h \varphi_1 + h^{k+s} \delta_0(x, 0), \quad D_t u_h(x, 0) = T_h \varphi_2 + h^{k+s} \delta_0(x, 0) \]

yields \((u_h - \bar{u})(0) = O(h^{k+s})\) in \(H^1\) norm, and \(D_t(u_h - \bar{u})(0) = O(h^{k+s})\) in \(L^2\) norm.

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