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On Maximal Finite Irreducible Subgroups of $GL(n, \mathbf{Z})$ I. The Five and Seven Dimensional Cases

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Abstract. General methods for the determination of maximal finite absolutely irreducible subgroups of $GL(n, \mathbf{Z})$ are described. For $n = 5, 7$ all these groups are computed up to $\mathbf{Z}$-equivalence.

1. Introduction. By the Jordan-Zassenhaus Theorem [5], [14] there is only a finite number of conjugacy classes of finite subgroups of $GL(n, \mathbf{Z})$, the group of all integral $n \times n$ matrices with determinant ± 1. For $n = 2, 3$ they were already classified in the last century. They are used for describing symmetry properties of crystals. Recently, the groups in four dimensions were determined in two steps. Firstly, E. C. Dade [6] found the maximal finite subgroups of $GL(4, \mathbf{Z})$ in 1965. Then R. Bülow, J. Neubüser, and H. Wondratschek [2] computed all finite subgroups by means of electronic computation. Later on the maximal finite subgroups of $GL(5, \mathbf{Z})$ were determined independently by S. S. Ryškov [11], [12] and R. Bülow [1]. Our aim is to give a list of the maximal finite irreducible subgroups of $GL(7, \mathbf{Z})$ which can be found in Section 6, Theorem (6.6). We also give a short derivation of the corresponding groups in five dimensions in Section 7. Other dimensions, e.g. $n = 6$, will follow later on.

Prior to the exposition of our method we remark:

(1) Let Δ be an irreducible integral representation of a finite group G. If the degree of Δ is an odd prime number, Δ is irreducible over $\mathbf{Z}$ if and only if Δ is irreducible over $\mathbf{C}$, i.e. every irreducible representation is absolutely irreducible [5]. Unless otherwise stated, irreducibility means always absolute irreducibility in the following.

(2) Every finite subgroup G of $GL(n, \mathbf{Z})$ fixes a positive definite integral $n \times n$ matrix X: $g^T X g = X$ for all $g \in G$. For instance, $X := \Sigma_{g \in G}\, g^T g$ will do. By Schur's Lemma X is uniquely determined up to scalar multiples in case G is irreducible, i.e. its natural representation Δ: $G \longrightarrow GL(n, \mathbf{Z})$: $g \longmapsto g$ is irreducible. On the other hand, the $\mathbf{Z}$-automorph of an integral positive definite matrix X is finite since it is a discrete subgroup of the compact $\mathbf{R}$-automorph. Hence the irreducible maximal finite subgroups G of $GL(n, \mathbf{Z})$ are the $\mathbf{Z}$-automorphs of each integral positive definite $n \times n$ matrix which is fixed by G, and G is uniquely determined by each of its irreducible subgroups.

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Therefore, we proceed in the following manner. We derive all minimal irreducible subgroups of $GL(n, \mathbf{Z})$ up to rational equivalence. So we must look for all finite groups which have a rational irreducible faithful representation of degree $n = 7$ such that this representation becomes reducible for every nontrivial subgroup. Next we compute the integral equivalence classes into which the rational equivalence classes of these representations are divided. This is done by means of the centering algorithm [10] which will be described in the next section. Together with the representations the fixed forms are computed also. In a last step we obtain the maximal irreducible subgroups of $GL(n, \mathbf{Z})$ as the $\mathbf{Z}$-automorphs of those forms.

We find seven groups up to integral equivalence. They are listed in Section 6.

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2. The Centering Algorithm. The algorithm was developed in [10] to find all integral classes into which the rational class of a given irreducible $\mathbf{Z}$-representation splits. We give a brief account of part of the results and describe the corresponding computer program.

Let G be a finite group and L a $\mathbf{Z}G$-representation module, i.e. a $\mathbf{Z}G$-module which is also a free abelian additive group of finite rank. By a centering of L we mean a $\mathbf{Z}G$-submodule of L of finite index in L or, equivalently, of the same rank as L. Two $\mathbf{Z}G$-modules say L, L' are called $\mathbf{Z}$-equivalent ($\mathbf{Q}$-equivalent), if L and L' are $\mathbf{Z}G$-isomorphic ($\mathbf{Q}L$ and $\mathbf{Q}L'$ are $\mathbf{Q}G$-isomorphic) [5]. Hence every centering of L is $\mathbf{Q}$-equivalent to L. If two $\mathbf{Z}G$-modules L and L' are $\mathbf{Q}$-equivalent, then there exists a centering L'' of L such that L' and L'' are $\mathbf{Z}$-equivalent. For let φ: $\mathbf{Q}L' \longrightarrow \mathbf{Q}L$ be a $\mathbf{Q}G$-isomorphism. Because L is of finite rank there is a natural number k such that $k\varphi(L') \subseteq L$. Clearly, ψ: $L' \longrightarrow k\varphi(L')$: $l \longmapsto k\varphi(l)$ is a $\mathbf{Z}G$-isomorphism and $k\varphi(L')$ is a centering of L. The $\mathbf{Q}$-equivalence class of a $\mathbf{Z}G$-module L splits into a finite number of $\mathbf{Z}$-equivalence classes (Jordan-Zassenhaus Theorem, [5], [14]). By our remarks a set of representatives of the $\mathbf{Z}$-classes can be chosen from the centerings of L.

To find such a set of representatives we define a partial ordering on the set $\mathfrak{z}(L)$ of all centerings of L: for $M, N \in \mathfrak{z}(L)$ let $M \prec N$, if there is a natural number k with $kN = M$. If $M \prec N$ for $M, N \in \mathfrak{z}(L)$, then M and N are $\mathbf{Z}$-equivalent. Every centering of L is contained in a uniquely determined $\prec$-maximal centering. For a proof let $M \in \mathfrak{z}(L)$ and $e_1, \ldots, e_n$ and $\alpha_1 e_1, \ldots, \alpha_n e_n$ a pair of compatible $\mathbf{Z}$-bases of L, M, respectively ($\alpha_i \in \mathbf{Z}$, $i = 1, \ldots, n$; $\alpha_i | \alpha_{i+1}$, $i = 1, \ldots, n-1$). The $\prec$-maximal centering $\bar{M}$, $M \prec \bar{M}$, is given by $\bar{M} = \alpha_1^{-1} M$. Note, that M is itself $\prec$-maximal if and only if $\alpha_1 = \pm 1$. This also implies the uniqueness. We always consider L itself to be $\prec$-maximal.

(2.1) THEOREM. *Let L be an (absolutely) irreducible $\mathbf{Z}G$-representation module. Then the $\prec$-maximal centerings of L form a set of representatives of the $\mathbf{Z}$-classes contained in the $\mathbf{Q}$-class of L.*

Proof. By our remarks it suffices to show that two $\prec$-maximal centerings

cannot be $\mathbf{Z}$-equivalent. Let, therefore, $\bar{M}$ and $\bar{N}$ be two $\prec$-maximal centerings of L and φ: $\bar{M} \longmapsto \bar{N}$ a $\mathbf{Z}G$-isomorphism. For all $g \in G$ and all $m \in \bar{M}$ we have $\varphi(gm) = g\varphi(m)$. Since L is an irreducible $\mathbf{Z}G$-module, there exists an $r \in \mathbf{Q}\backslash\{0\}$, $\varphi(m) = rm$ by Schur's Lemma. This implies $N = rM \subseteq L$. From the $\prec$-maximality of M we derive $r \in \mathbf{Z}$ by choosing a pair of consistent bases for M and L. But then r has to be ± 1 because otherwise N would not be $\prec$-maximal. Q.E.D.

Now let M be a centering of L. Then the $\mathbf{Z}G$-composition factors of L/M are irreducible $\mathbf{Z}_pG$-representation modules for prime numbers p dividing the index of M in L. Let

$$0 = M_0/M \subset M_1/M \subset \cdots \subset M_{k-1}/M \subset M_k/M \quad (k \in \mathbf{N}, M_0 = M, M_k = L)$$

a G-composition series of L/M. Then the M_i $(i = 1, \ldots, k-1)$ are also centerings of L. In our algorithm we will obtain M_{i-1} from M_i $(i = k, k-1, \ldots, 1)$ as the kernel of a $\mathbf{Z}G$-epimorphism of M_i onto an irreducible $\mathbf{Z}_pG$-module for suitable p. Which irreducible $\mathbf{Z}_pG$-modules can actually occur as such composition factors?

Let $p_1, \ldots, p_s$ $(s \in \mathbf{N})$ be the different prime numbers dividing the index of M in L.

(2.2) LEMMA. *The irreducible $\mathbf{Z}_pG$-composition factors $(p \in \{p_1, \ldots, p_s\})$ of L/M occur already as $\mathbf{Z}_pG$-composition factors of L/pL.*

Proof. Let K be an arbitrary centering of L. By a theorem of Brauer and Nesbitt [5, p. 585] the irreducible $\mathbf{Z}_qG$-composition factors of L/qL and K/qK are the same for every prime number q. Let M_i/M_{i-1} be a composition factor of L/M $(1 \leqslant i \leqslant k)$ and $p \mid [M_i : M_{i-1}]$ $(p \in \{p_1, \ldots, p_s\})$. Then M_i/M_{i-1} is also a composition factor of M_i/pM_i, hence of L/pL. Q.E.D.

(2.3) LEMMA. *If M is a $\prec$-maximal centering of L, then the irreducible $\mathbf{Z}_pG$-composition factors of L/M have a $\mathbf{Z}_p$-rank which is smaller than the $\mathbf{Z}$-rank of L $(p \in \{p_1, \ldots, p_s\})$.*

Proof. Assume M_i/M_{i-1} is a $\mathbf{Z}_pG$-composition factor of L/M of $\mathbf{Z}_p$-rank equal to the $\mathbf{Z}$-rank of L $(i \in \{1, \ldots, k\}, p \in \{p_1, \ldots, p_s\})$. Hence $p^{-1}M_{i-1} = M_i \subseteq L$, implying $p^{-1}M \subseteq L$. But this is a contradiction to the $\prec$-maximality of M. Q.E.D.

Therefore, the index of a $\prec$-maximal centering M of L in L is divisible at most by those prime numbers p for which L/pL is reducible. A useful criterion for the irreducibility of L/pL is given in [5]:

(2.4) LEMMA. *Let p be a prime number. If the order of the p-Sylow group of G divides the $\mathbf{Z}$-rank of L, then L/pL is an irreducible $\mathbf{Z}_pG$-module.*

Hence in order to find the $\prec$-maximal centerings of L, we need only consider those prime numbers which divide the order of G.

The centering algorithm works as follows. By M we always denote the $\prec$-maximal centering of our given irreducible $\mathbf{Z}G$-representation module L.

(α) Let $M = L$.

(β) By solving systems of linear equations compute all centerings M_i of L with the following properties:

(i) $M_i \subset M$,

(ii) M_i is $\prec$-maximal in M,

(iii) M_i is maximal in M.

This step requires the knowledge of the G-composition factors of L/pL for each prime number p dividing the order of G.

(γ) For each M_i obtained in (β) check whether M_i is $\prec$-maximal in L. In this case store M_i. If M_i is not $\prec$-maximal in L, look for the previously obtained $\prec$-maximal centering of L which is equal to $p^{-1}M_i$ where p is the unique prime number dividing the index of M_i in M.

(δ) Take the next centering out of the storage for which (β) was not yet carried out, call it M, and go back to (β). If no such M is available, the algorithm terminates.

Now let us describe the details necessary for the execution of the algorithm. Let $G = \langle g_1, \ldots, g_r \rangle$ and Δ: $G \longrightarrow GL(n, \mathbf{Z})$ be an irreducible representation of G with the corresponding $\mathbf{Z}G$-representation module $L = \mathbf{Z}^{n \times 1}$. Let $p_1, \ldots, p_k$ be all prime numbers for which L/p_iL becomes reducible ($i = 1, \ldots, k$). By Δ_{ij} we denote the irreducible $\mathbf{Z}_{p_i}$-constituents of the representation belonging to L/p_iL with j running from 1 to $s(i)$. The corresponding degrees n_{ij} are smaller than n.

As input data we have

$$n, r, \Delta(g_1), \Delta(g_2), \ldots, \Delta(g_r);$$

for $i = 1, \ldots, k$: p_i, $s(i)$, n_{ij}, $\Delta_{ij}(g_1)$, $\Delta_{ij}(g_2)$, $\ldots$, $\Delta_{ij}(g_r)$ $(1 \leqslant j \leqslant s(i))$.

The output consists of bases of all $\prec$-maximal centerings M_ν of L (expressed in the standard basis of L) in the form of an integral $n \times n$-matrix U_ν, further of U_ν^{-1}, the invariant factors of U_ν, and $\Delta_\nu(g_\mu) := U_\nu^{-1}\Delta(g_\mu)U_\nu$ $(\mu = 1, \ldots, r)$ with Δ_ν denoting the integral representation of G belonging to M_ν. (The Δ_ν form a full set of representatives of all integral representations of G which are $\mathbf{Q}$-equivalent to Δ.) ν runs from 1 to $h = h(\Delta)$, the class number of Δ. The output also provides the lattice of centerings. The computer program is composed of the following steps:

(2.5) (1) $M = L$, $\Delta_M = \Delta$, store L (i.e. I_n, the unit matrix representing a basis of L).

(2) for every prime number p_i $(i = 1, \ldots, k)$ and the corresponding constituents Δ_{ij} $(j = 1, \ldots, s(i))$ proceed as follows.

(3) Determine all solutions $\varphi \in (\mathbf{Z}_{p_i})^{n_{ij} \times n}$ of the system of linear equations $\varphi\Delta_M(g_\mu) = \Delta_{ij}(g_\mu)\varphi$ $(\mu = 1, \ldots, r)$. Each nontrivial solution represents a G-epimorphism from M onto an irreducible finite G-module, the kernel of which is a maximal centering of M. If two nontrivial solutions are linearly dependent, they yield the same centering. If there is no nontrivial solution, go back to (2).

(4) Choose one vector $\varphi \neq 0$ out of every one dimensional subspace of the space of all solutions of (2) and go on.

(5) Compute a $\mathbf{Z}$-basis of the centering N of all $x \in \mathbf{Z}^{n \times 1}$ satisfying $\varphi x = 0$, express it in terms of the basis of L, and determine the invariant factors of the attached matrix.

(6) If the greatest common divisor of the invariant factors is greater than one, go to (8).

(7) Check, whether the computed new centering N is identical to one already stored. If it is not, store it. Go back to (4).

(8) Determine the stored $\prec$-maximal centering which is $\mathbf{Z}$-equivalent to N that is identical to $p_i^{-1}N$. Go back to (4).

(9) Take the next centering M (that is the matrix U of its basis) out of the storage, for which (2) was not yet carried out. Compute $\Delta_M(g_\mu) = U^{-1}\Delta(g_\mu)U$ ($\mu = 1, \ldots, r$) and go back to (2). If no centering M is left, the program terminates.

Remarks. (2.6) If some constituent Δ_{ij} of the modular representation belonging to L/p_iL is not absolutely irreducible, the algorithm can be shortened at step (4). Instead of choosing a vector $\varphi \neq 0$ out of every one dimensional subspace it suffices to choose one out of certain m_{ij}-dimensional subspaces with trivial intersections, where m_{ij} denotes the dimension of the commuting algebra of Δ_{ij}.

(2.7) Clearly, every $\prec$-maximal centering of L is the intersection of uniquely determined centerings with prime power index in L. This can be used to modify the algorithm in the way that one computes the $\prec$-maximal centerings of prime power index for each prime p_i ($i = 1, \ldots, k$) separately and forms the intersections afterwards.

(2.8) As we mentioned in the introduction, we are also interested in the quadratic forms which are fixed by $\Delta_\nu(G)$, i.e. we compute the symmetric matrices $X_\nu \in \mathbf{Z}^{n\times n}$ satisfying $\Delta_\nu(g)^T X_\nu \Delta_\nu(g) = X_\nu$ for all $g \in G$ and $1 \leqslant \nu \leqslant h(\Delta)$. Since X_ν is unique up to scalar multiples, it suffices to determine one X_ν for every $\nu = 1, \ldots, h(\Delta)$. Let $\Delta_1 = \Delta$, then $X_\nu = U_\nu^T X_1 U_\nu$. This computation can easily be implemented into the computer program.

3. Preliminary Considerations About Integral Representations. Since we want to determine the minimal irreducible finite subgroups of $GL(n, \mathbf{Z})$ up to rational equivalence for certain n the following version of Clifford's Theorem [5] will be useful.

(3.1) THEOREM. *Let G be a finite group, $\Delta\colon G \longrightarrow GL(n, \mathbf{Z})$ be an irreducible (not necessarily absolutely irreducible) representation of G, and N a normal subgroup of G.*

(1) *There exist natural numbers k, r, m with $n = krm$ and r rationally inequivalent $\mathbf{Q}$-irreducible integral representations $\Delta_1, \ldots, \Delta_r$ of N, all of the same degree m, which satisfy $\Delta_1(N) = \cdots = \Delta_r(N)$; and the restriction $\Delta|_N$ is rationally equivalent to $k(\Delta_1 \dotplus \cdots \dotplus \Delta_r)$.*

(2) *There exists a representation Δ' of G, which is rationally equivalent to Δ, and $\Delta'|_N = \Gamma_1 \dotplus \cdots \dotplus \Gamma_r$, where $\Gamma_1, \ldots, \Gamma_r$ are integral representations of N satisfying $\Gamma_i \sim_{\mathbf{Q}} k\Delta_i$ ($i = 1, \ldots, r$) and $\Gamma_1(N) = \cdots = \Gamma_r(N)$.*

Proof. (1) This is essentially Clifford's Theorem on the restriction of irreducible representations to normal subgroups. That the Δ_i ($i = 1, \ldots, r$) can be chosen integral follows from [5, Theorem 73.5].

(2) Let V be a $\mathbf{Q}G$-representation module belonging to Δ and U the inertia group of Δ_1, $U := \{g \in G \mid \Delta_1^g \sim_{\mathbf{Q}} \Delta_1\}$, where $\Delta_1^g\colon G \longrightarrow GL(m, \mathbf{Z})\colon h \longmapsto \Delta_1(ghg^{-1})$. Let $\overline{V}$ be an irreducible $\mathbf{Q}U$-submodule of V. If $g_1, \ldots, g_r$ form a system of (right) representatives of U in G, then $V = g_1\overline{V} \oplus \cdots \oplus g_r\overline{V}$ is a direct sum of the irreduc-

ible $\mathbf{Q}U$-modules $g_i\bar{V}$ $(i = 1, \ldots, r)$ by Clifford's theory. By [5, Theorem 73.5] $\bar{V}$ contains a $\mathbf{Z}U$-module $\widetilde{V}$ with $\mathbf{Q}\widetilde{V} = \bar{V}$. Clearly, $g_1\widetilde{V} \oplus \cdots \oplus g_r\widetilde{V}$ is a $\mathbf{Z}G$-representation module. Let Δ' be the corresponding integral representation of G. Then $\Delta' \sim_{\mathbf{Q}} \Delta$ holds and $\Delta'|_N = \Gamma_1 \dotplus \cdots \dotplus \Gamma_r$ where the $\mathbf{Z}N$-module for Γ_i is given by $g_i\widetilde{V}$ for $i = 1, \ldots, r$. Q.E.D.

The $\mathbf{Z}$-automorph of the unit matrix I_n is called the full monomial group H_n of degree n. Clearly, H_n is irreducible and, therefore, a maximal finite irreducible subgroup of $GL(n, \mathbf{Z})$. To determine certain minimal irreducible subgroups of H_n the following theorem is useful:

(3.2) THEOREM. *Let G be a minimal irreducible subgroup of H_n with natural representation Δ, and φ: $G \longrightarrow \widetilde{S}_n$: $(g_{ij}) \longrightarrow (|g_{ij}|)$ a homomorphism of G into the group $\widetilde{S}_n$ of all permutation matrices of degree n. If $\Delta|_{\ker\varphi} = \Delta_1 \dotplus \cdots \dotplus \Delta_n$ with pairwise inequivalent representations Δ_i $(i = 1, \ldots, n)$, $\varphi(G)$ yields a minimal transitive permutation group of degree n, that is a transitive permutation group which has no proper transitive subgroups. If G splits over* ker φ, *then* ker φ *is characterized by the following three properties*:

(i) ker $\varphi \subseteq \{\operatorname{diag}(a_1, \ldots, a_n) | a_i = \pm 1\ (i = 1, \ldots, n)\}$, *and* ker φ *is invariant under conjugation by $\varphi(G)$.*

(ii) *The projections Δ_i*: ker $\varphi \longrightarrow \{\pm 1\}$: $\operatorname{diag}(a_1, \ldots, a_n) \longrightarrow a_i$ $(i = 1, \ldots, n)$ *are pairwise unequal.*

(iii) ker φ *is minimal with the properties* (i) *and* (ii).

Proof. Since the Δ_i $(i = 1, \ldots, n)$ are pairwise inequivalent, Δ is irreducible if and only if $\varphi(G)$ is transitive. Hence, if G is minimal irreducible, $\varphi(G)$ is minimal transitive. For $g \in G$ and $a_i = \pm 1$ $(i = 1, \ldots, n)$ we have $g \cdot \operatorname{diag}(a_1, \ldots, a_n)g^{-1} = \varphi(g) \operatorname{diag}(a_1, \ldots, a_n)\, \varphi(g)^{-1}$. The characterization of ker φ follows immediately. Q.E.D.

4. The Minimal Irreducible Finite Subgroups of $GL(7, \mathbf{Z})$. First we consider the solvable case.

(4.1) THEOREM. *Let G be a minimal irreducible finite subgroup of $GL(7, \mathbf{Z})$. If G is solvable, G is $\mathbf{Q}$-equivalent to*

$$G_1 = \left\langle g_1 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \right\rangle.$$

G is isomorphic to $A(1, 8)$ the full affine group on $\mathbf{F}_8$, the galois field of eight elements.

Proof. Let N be a maximal abelian normal subgroup of G and Δ the natural

representation of G. Applying Theorem (3.1), we get three natural numbers k, r, m, $7 = krm$ and a decomposition of Δ in the form $\Delta|_N = \Gamma_1 \dotplus \cdots \dotplus \Gamma_r$ (w.l.o.g. $\Delta = \Delta'$). The assumption $m = 7$ leads to a contradiction:

In this case $N = \Delta(N)$ is an irreducible abelian group. Then the enveloping algebra $E(N)$ of N is a simple commutative $\mathbf{Q}$-algebra. By Wedderburn's theorem $E(N)$ is a field. Because N is finite, $E(N)$ is a cyclotomic field of degree 7 over $\mathbf{Q}$. But the degrees of the cyclotomic fields are given by the values of the Euler-φ-function which are even or equal to 1. Therefore, m has to be 1. The assumption $k = 7$ also leads to a contradiction: $k = 7$ implies $N = \langle I_7 \rangle$ or $N = \langle -I_7 \rangle$, hence N is contained in the center of G. So N cannot be a maximal abelian subgroup since G is solvable. Thus, $\Delta|_N = \Gamma_1 \dotplus \cdots \dotplus \Gamma_7 = \Delta_1 \dotplus \cdots \dotplus \Delta_7$ with pairwise inequivalent representations Δ_i $(i = 1, \ldots, 7)$, and the premises of Theorem (3.2) are fulfilled. As a minimal transitive permutation group we can choose $\langle (1\ 2\ 3\ 4\ 5\ 6\ 7) \rangle$, which is the only one up to equivalence by Sylow's theorem, and get $\varphi(G) = \langle g_2 \rangle$. $\varphi(G)$ operates on $D := \{\operatorname{diag}(a_1, \ldots, a_7) | a_i = \pm 1\ (i = 1, \ldots, 7)\}$ by conjugation. D is a $\mathbf{Z}_2$-vector space, and g_2 induces a linear transformation with the characteristic polynomial $x^7 - 1 = (x + 1)(x^3 + x^2 + 1)(x^3 + x + 1)$. The corresponding decomposition of D as a $\varphi(G)$-module is given by $D = N_1 \oplus N_2 \oplus N_3$ with irreducible $\mathbf{Z}_2\varphi(G)$-modules

$$N_1 = \langle \operatorname{diag}(-1,-1,-1,-1,-1,-1,-1) \rangle,$$

$$N_2 = \langle \operatorname{diag}(-1,-1,-1,1,-1,1,1), \operatorname{diag}(1,-1,-1,-1,1,-1,1), \operatorname{diag}(1,1,-1,-1,-1,1,-1) \rangle,$$

$$N_3 = \langle \operatorname{diag}(-1,1,-1,-1,-1,1,1), \operatorname{diag}(1,-1,1,-1,-1,-1,1), \operatorname{diag}(1,1,-1,1,-1,-1,-1) \rangle,$$

Hence we find $\ker \varphi$ is equal to N_2 or N_3.

By the Schur-Zassenhaus Theorem N has a complement in G which can be chosen to be generated by g_2 (after conjugation by a monomial matrix). So we get two groups $\langle N_2, g_2 \rangle$, $\langle N_3, g_2 \rangle$, which turn out to be rationally equivalent. Indeed, an easy computation shows that both groups are isomorphic to the full affine group on $\mathbf{F}_8$, which has only one irreducible representation of degree 7. Q.E.D.

For the derivation of the nonsolvable groups the following lemma is useful.

(4.2) LEMMA. *Let G be a minimal irreducible finite subgroup of $GL(n, \mathbf{Z})$, n odd. Then G is already contained in $SL(n, \mathbf{Z})$. In particular, the center of G is trivial.*

Proof. This is an easy application of Clifford's theory. The last statement then follows from Schur's Lemma and $\det(-I_n) = -1$ for odd n. Q.E.D.

(4.3) THEOREM. *The minimal irreducible finite subgroups G of $GL(7, \mathbf{Z})$, which are imprimitive as subgroups of $GL(7, \mathbf{C})$, are solvable or isomorphic to $PSL(2, 7)$, which in fact is isomorphic to a subgroup of H_7.*

Proof. Because 7 is a prime number G is $\mathbf{C}$-equivalent to a complex monomial group. Let G be given in this form, and let

$$\varphi\colon G \longrightarrow \widetilde{S}_7\colon (g_{ij}) \longmapsto (\rho(g_{ij})), \qquad \rho(g_{ij}) = \begin{cases} 0 & \text{for } g_{ij} = 0, \\ 1 & \text{otherwise,} \end{cases}$$

be the associated permutation representation of the natural representation Δ of G. The restriction of Δ to $\ker \varphi$ decomposes into seven one dimensional representations: $\Delta|_{\ker \varphi} = \Delta_1 + \cdots + \Delta_7$. By Clifford's theory the Δ_i $(i = 1, \ldots, 7)$ are either all pairwise inequivalent or all equal. In the first case we conclude as in (3.2) and (4.1) that $\varphi(G)$ is isomorphic to the cyclic group of order 7 and hence G is solvable. In the second case $\ker \varphi \leqslant \langle -I_7 \rangle$ holds, but because of (4.2) φ has to be injective and so G is isomorphic to a transitive permutation group of degree 7. The transitive permutation groups of degree 7 are either solvable or isomorphic to one of the groups $PSL(2, 7)$, A_7, or S_7. Among these only $PSL(2, 7)$ has an irreducible representation of degree 7. The corresponding linear group can be chosen as a subgroup of H_7. Since the orders of the proper subgroups of $PSL(2, 7)$ are smaller than 50, it is a minimal irreducible group. Q.E.D.

The primitive finite subgroups of $SL(7, \mathbf{C})$ were determined by D. B. Wales in [13]. From his results we conclude

(4.4) THEOREM. *There are no minimal irreducible finite subgroups G of $GL(7, \mathbf{Z})$ which are primitive (as subgroups of $GL(7, \mathbf{C})$).*

Proof. Because of (4.2) all groups G, which we are interested in, must be contained in Wales' list. By a result of Minkowski [9] the order of a finite subgroup of $GL(7, \mathbf{Z})$ divides $2^{11}3^{4}5 \cdot 7$. Thus G must be isomorphic to one of the following five isomorphism types:

(I) $PSL(2, 8)$, (II) A_8, (III) $PSL(2, 7)$, (IV) $PSU(3, 9)$, (V) $S_6(2)$.

Ad(I). $PSL(2, 8)$ contains $A(1, 8)$, the affine group on $\mathbf{F}_8$. $A(1, 8)$ has exactly one faithful representation of degree 7, and this representation is irreducible ((4.1)). Hence there is no minimal irreducible finite subgroup of $GL(7, \mathbf{Z})$ isomorphic to $PSL(2, 8)$.

Ad(II). In this case the same argument as in (I) applies.

Ad(III). Compare Theorem (4.3).

Ad(IV). $PSU(3, 9)$ has three irreducible representations of degree 7, only one of them can be made rational. (For a character table see [7].) From the representation theory of $PSL(2, 7)$ one can easily conclude that $PSU(3, 9)$ has a subgroup which is isomorphic to $PSL(2, 7)$. An inspection of the character tables of both groups shows that G ($\cong PSU(3, 9)$) is not minimal irreducible.

Ad(V). In Section 6 we shall see that the group $S_6(2)$ has a subgroup, which is isomorphic to the symmetric group S_8. This subgroup has only two faithful representations of degree 7 both being irreducible. Hence similar to (I) we conclude, that G ($\cong S_6(2)$) is not minimal irreducible. Q.E.D.

5. Computation of the Z-Classes. We proved in the last section that there are exactly two minimal irreducible finite subgroups of $GL(7, \mathbf{Z})$ up to rational equivalence. As representatives, we choose $G_1 \cong A(1, 8)$, given in (4.1), and $G_2 \cong PSL(2, 7)$, given

in the form

$$G_2 = \left\langle g_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \right.$$

$$\left. g_3 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix} \right\rangle.$$

An isomorphism of G_2 onto $PSL(2, 7)$ is:

$$g_1 \longrightarrow \pm\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad g_3 \longrightarrow \pm\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The form fixed by G_1 and G_2 is the unit form in seven variables. As input data we also need the modular constituents of the natural representations Δ_1, Δ_2 of G_1, G_2, respectively. Both representations become reducible only modulo 2. For G_1 the 2-modular constituents of Δ_1 are:

$$\Delta_{11}\colon G_1 \longrightarrow \mathbf{Z}_2^\times\colon g \longmapsto 1;$$

$$\Delta_{12}\colon G_1 \longrightarrow GL(3, \mathbf{Z}_2)\colon g_2 \longmapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad g_1 \longmapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix};$$

$$\Delta_{13}\colon G_1 \longrightarrow GL(3, \mathbf{Z}_2)\colon g_2 \longmapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad g_1 \longmapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(Note: Δ_{12}, Δ_{13} are not absolutely irreducible.)

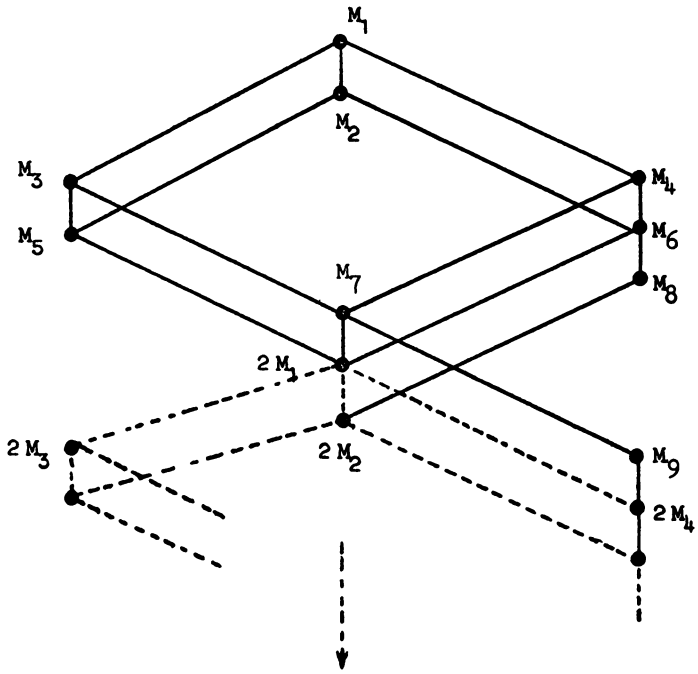
For G_2 and Δ_2 we get:

$$\Delta_{21}\colon G_2 \longrightarrow \mathbf{Z}_2^\times\colon g \longmapsto 1,$$

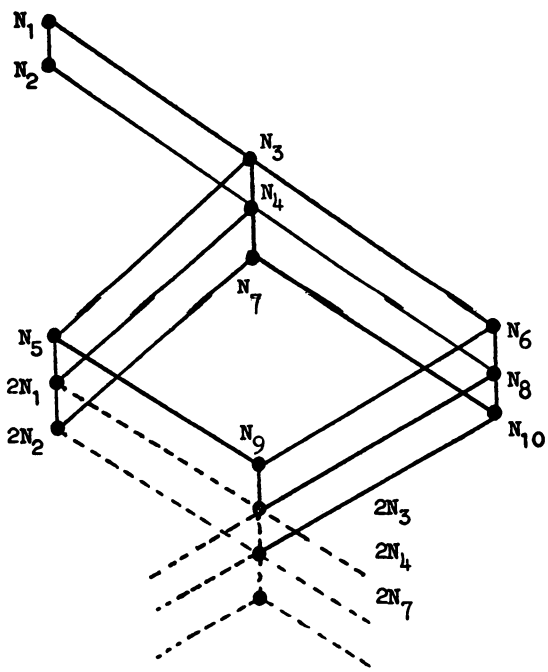
$$\Delta_{22}\colon G_2 \longrightarrow GL(3, \mathbf{Z}_2)\colon g_1 \longmapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad g_3 \longmapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix};$$

$$\Delta_{23}\colon G_2 \longrightarrow GL(3, \mathbf{Z}_2)\colon g_1 \longmapsto \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad g_3 \longmapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

The lattice of centerings for G_1 has the form



and for G_2



The bases of the centerings M_i, N_i expressed in the basis of M_1, N_1, respectively, are the columns of the following matrices A_i, B_i. They are ordered according to the corresponding forms $\lambda A_i^T A_i$, $\lambda B_i^T B_i$, respectively, ($\lambda \in \mathbf{Q}$).

Quadratic form: $F_1 = I_7$.

Bases of corresponding centerings: $A_1 = B_1 = I_7$ and

$$B_8 = (x, g_1x, g_1^2x, \ldots, g_1^6x) \quad \text{where } x^T = (0, 1, 1, -1, 0, 0, 1).$$

Quadratic form:

$$F_2 = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 2 \end{pmatrix}.$$

Bases of corresponding centerings:

$$A_2 = B_2 = (x, g_1x, \ldots, g_1^5x, y) \quad \text{with } x^T = (1, 1, 0, \ldots, 0),$$

$$y^T = (0, \ldots, 0, 1, -1) \quad \text{and} \quad B_{10} = B_8B_2.$$

Quadratic form:

$$F_3 = \begin{pmatrix} 4 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 4 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 4 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 4 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 4 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 4 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 7 \end{pmatrix}.$$

Bases of corresponding centerings:

$$A_7 = B_6 = (x, g_1x, \ldots, g_1^5x, y) \quad \text{with } x^T = (2, 0, \ldots, 0),$$

$$y^T = (1, \ldots, 1) \quad \text{and} \quad B_5 = \tfrac{1}{2}B_8B_6.$$

Quadratic form: $F_4 = 8I_7 - J_7$ where $J_7 \in \mathbf{Z}^{7\times 7}$ has all its entries equal to 1.

Bases of corresponding centerings:

$$A_9 = (x, g_1x, \ldots, g_1^6x) \quad \text{with } x^T = (-1, -1, -1, 1, -1, 1, 1) \quad \text{and}$$

$$B_9 = (y, g_1y, \ldots, g_1^6y) \quad \text{with } y^T = (-1, 1, 1, -1, 1, -1, -1).$$

Quadratic form: $F_5 = I_7 + J_7$.

Bases of corresponding centerings:

$$A_8 = (x, g_1x, \ldots, g_1^6x) \quad \text{with } x^T = (1, 0, 0, 1, 1, 1, 0) \quad \text{and}$$

$$B_7 = (y, g_1y, \ldots, g_1^6y) \quad \text{with } y^T = (1, 1, 0, 0, 1, 0, 1).$$

Quadratic form:

$$F_6 = \begin{pmatrix} 2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}.$$

Bases of corresponding centerings:

$$B_4 = A_5 = \begin{pmatrix} 2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 \end{pmatrix} \quad \text{and} \quad A_6 = \begin{pmatrix} 2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & 1 & 0 & -1 & -1 & 1 \\ 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

Quadratic form:

$$F_7 = \begin{pmatrix} 3 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 3 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 3 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 3 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & 3 & -1 \\ 1 & 1 & 1 & 1 & 1 & -1 & 3 \end{pmatrix}.$$

Bases of corresponding centerings:

$A_3 = B_3 = (x, g_1x, \ldots, g_1^4x, y, g_1^6x)$ with $x^T = (1, 1, 0, 1, 0, 0, 0)$,

$y^T = (0, 1, 0, 0, 0, -1, -1)$ and $B_4 = (\bar{x}, g_1\bar{x}, \ldots, g_1^4\bar{x}, \bar{y}, g_1^6\bar{x})$ with

$\bar{x}^T = (1, 0, 1, 1, 0, 0, 0)$, $\bar{y}^T = (0, 0, 0, 1, 0, -1, -1)$.

6. The Irreducible Maximal Finite Subgroups of $GL(7, \mathbf{Z})$. Let $\Delta: G \longrightarrow GL(n, \mathbf{Z})$ be a representation of the group G. By Δ^{-T} we denote the inverse transposed representation of Δ defined by $\Delta^{-T}(g) = \Delta(g^{-1})^T$ for all $g \in G$. The following elementary theorem will be useful [10]:

(6.1) THEOREM. *Let G be a finite group with representation $\Delta: G \longrightarrow GL(n, \mathbf{Z})$. Let $X \in \mathbf{Q}^{n \times n}$ be symmetric and nonsingular.*

(i) *Δ^{-T} is $\mathbf{Q}$-equivalent to Δ.*

(ii) *X is fixed by $\Delta(G)$ if and only if X^{-1} is fixed by $\Delta^{-T}(G)$.*

(iii) *The $\mathbf{Z}$-automorph of X is $\mathbf{Q}$-equivalent to the $\mathbf{Z}$-automorph of X^{-1}.*

Proof. We already remarked in the introduction that there is a symmetric positive definite matrix $Y \in \mathbf{Z}^{n \times n}$ satisfying $\Delta(g)^T Y \Delta(g) = Y$ for all $g \in G$. This equation implies $\Delta^{-T}(g) = Y\Delta(g)Y^{-1}$ for all $g \in G$, which proves (i). By inverting both sides of the equation $\Delta(g)^T X \Delta(g) = X$ ($g \in G$) we get (ii). Then (iii) follows immediately. Q.E.D.

Assume that M is a $\prec$-maximal centering of the irreducible $\mathbf{Z}G$-representation module L and that Δ is the according representation of M. For the $\prec$-maximal centering of L belonging to Δ^{-T} we write $M^{\#}$. Clearly, $(M^{\#})^{\#} = M$.

For the centerings M_i ($i = 1, \ldots, 9$) of G_1 derived in Section 5 an easy computation shows $M_1^{\#} = M_1$, $M_2^{\#} = M_7$, $M_3^{\#} = M_5$, $M_4^{\#} = M_6$, $M_8^{\#} = M_9$. For G_2 we get $N_1^{\#} = N_1$, $N_2^{\#} = N_5$, $N_3^{\#} = N_4$, $N_6^{\#} = N_{10}$, $N_7^{\#} = N_9$, $N_8^{\#} = N_8$. Therefore, F_1^{-1} is

$\mathbf{Z}$-equivalent to F_1, $2F_2^{-1} \sim_{\mathbf{Z}} F_3$ ($\det(F_2) = 4$), $8F_4^{-1} \sim_{\mathbf{Z}} F_5$ ($\det(F_4) = 8^6$), $2F_6^{-1} \sim_{\mathbf{Z}} F_7$ ($\det(F_6) = 2$). Now we can determine all irreducible maximal finite subgroups of $GL(7, \mathbf{Z})$. Namely they are $\mathbf{Z}$-equivalent to the automorphs of one of the quadratic forms $F_1, \ldots, F_7$. We already mentioned that the automorph of F_1 is the full monomial group H_7 of degree 7. Moreover, the following two theorems hold. They are special cases of (III.6) and (III.3) in [10].

(6.2) THEOREM. *The automorphs of F_1, F_2, and F_3 are $\mathbf{Q}$-equivalent maximal finite subgroups of $GL(7, \mathbf{Z})$.*

Proof. By (6.1) the automorphs of F_2 and F_3 are $\mathbf{Q}$-equivalent. Clearly,

$$\left\{(a_1, \ldots, a_7)^T \mid a_i \in \mathbf{Z}, \sum_{i=1}^{7} a_i \equiv 0 \bmod 2\right\} \leqslant \mathbf{Z}^{7\times 1}$$

and

$$\{(a_1, \ldots, a_7)^T \mid a_i \in \mathbf{Z}, a_i \equiv a_j \bmod 2\ (i, j = 1, \ldots, 7)\} \leqslant \mathbf{Z}^{7\times 1}$$

are centerings of H_7 and correspond to the forms F_2, F_3, respectively. Hence H_7 is rationally equivalent to subgroups of the automorphs of F_2, F_3 which are in fact equal to the automorphs. The last statement follows easily since the vectors of shortest length in the second lattice are of the form $(0, \ldots, 0, \pm 2, 0, \ldots, 0)$. Q.E.D.

(6.3) THEOREM. *The automorphs of F_4 and F_5 are $\mathbf{Q}$-equivalent maximal finite subgroups of $GL(7, \mathbf{Z})$. They are isomorphic to $C_2 \times S_8$.*

Proof. Define a faithful $\mathbf{Z}S_8$-representation module $L = \bigoplus_{i=1}^{8} \mathbf{Z}e_i$ by $\pi e_i = e_{\pi(i)}$ for all $\pi \in S_8$ ($i = 1, \ldots, 8$), which corresponds to the natural permutation representation of S_8. Clearly, $L' := \mathbf{Z}(\Sigma_{i=1}^{8} e_i)$ is an invariant submodule and L/L' is a faithful irreducible $\mathbf{Z}S_8$-representation module. We write $\bar{e}_i = e_i + L'$ ($i = 1, \ldots, 8$). $\bar{e}_1, \ldots, \bar{e}_7$ constitute a basis of L/L' and $\Sigma_{i=1}^{8} \bar{e}_i = 0$ holds. The scalar product Φ: $\Phi(\bar{e}_i, \bar{e}_j) = 8\delta_{ij} - 1$ ($i, j = 1, \ldots, 7$) is invariant under S_8. The shortest vectors of this lattice are $\pm\bar{e}_1, \ldots, \pm\bar{e}_8$. This implies that the $\mathbf{Z}$-automorph of Φ is isomorphic to $\langle -\mathrm{id}\rangle \times S_8$, and the theorem follows by (6.1). Q.E.D.

In order to deal with the last two forms we need

(6.4) THEOREM. *The automorphs of F_6 and F_7 are $\mathbf{Q}$-equivalent maximal finite subgroups of $GL(7, \mathbf{Z})$. They are isomorphic to the Weyl group $W(E_7)$ of order $2^{10}3^4 5 \cdot 7$.*

Proof. The automorphs are $\mathbf{Q}$-equivalent by (6.1). The form F_6 can also be derived from the root system E_7 [8, p. 66], and the automorph of F_6 is equal to the automorphism group of the root system, in this case the Weyl group $W(E_7)$. Q.E.D.

In order to complete the proof of (4.4) we still have to verify the

(6.5) *Remark.* The Weyl group $W(E_7) \cong C_2 \times S_6(2)$ contains a subgroup isomorphic to S_8, hence $S_6(2)$ has a subgroup isomorphic to S_8.

Proof. In the terminology of the proof of (6.3) the $K_\nu = \{\Sigma_{i=1}^{8} a_i\bar{e}_i \mid a_i \in \mathbf{Z}, \Sigma_{i=1}^{8} a_i \equiv 0 \bmod \nu\}$ for $\nu = 1, 2, 4, 8$ are centerings of L/L' ($K_1 = L/L'$). From K_4 we obtain the form F_6 (from K_2 the form F_7, from K_8 the form F_5). Hence the automorph of F_4, which is isomorphic to $C_2 \times S_8$, is rationally equivalent to a subgroup of the automorph of F_6 (equal to $W(E_7)$). Q.E.D.

Therefore we have proved:

(6.6) THEOREM. *There are exactly seven irreducible maximal finite subgroups of $GL(7, \mathbf{Z})$ up to $\mathbf{Z}$-equivalence. The automorphs of the quadratic forms $F_1, \ldots, F_7$ as given in Section 5 form a full set of representatives. (For a description of these groups see Theorems* (6.2) *to* (6.4).)

7. **The Maximal Irreducible Finite Subgroups of $GL(5, \mathbf{Z})$.** The results for the five dimensional case can be obtained very easily by our method. In analogy to (4.1) we find only one solvable minimal irreducible finite subgroup G_1 of $GL(5, \mathbf{Z})$, up to $\mathbf{Q}$-equivalence:

$$G_1 = \left\langle g_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad g_2 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right\rangle.$$

By checking R. Brauer's list of the irreducible finite subgroups of $SL(5, \mathbf{C})$ [3] we see that there is only one nonsolvable group G_2 up to $\mathbf{Q}$-equivalence. (Note that the order of such a group has to divide $2^8 3^2 5$ [9].) G_2 is isomorphic to the alternating group A_5 and is obtained by reduction of the doubly transitive permutation representation of A_5 of degree 6.

$$G_2 = \left\langle g_1, g_3 = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{pmatrix} \right\rangle.$$

An isomorphism of G_2 onto A_5 is:

$$g_1 \longmapsto (1\ 2\ 3\ 4\ 5), \qquad g_2 \longmapsto (1\ 2)(3\ 4).$$

The forms fixed by G_1, G_2 are I_5, $6I_5 - J_5$, respectively. The natural representations Δ_1, Δ_2 of G_1, G_2 have the modular constituents:

Δ_1 becomes reducible only modulo 2:

$$\Delta_{11}: G_1 \longrightarrow \mathbf{Z}_2^{\times}: g \longmapsto 1;$$

$$\Delta_{12}: G_1 \longrightarrow GL(4, \mathbf{Z}_2): g_1 \longmapsto \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad g_2 \longmapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Δ_2 becomes reducible modulo 2 and modulo 3:

$\Delta_{21}\colon G_2 \longrightarrow \mathbf{Z}_2^\times\colon g \longmapsto 1;$

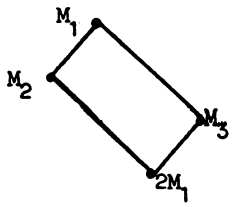
$$\Delta_{22}\colon G_2 \longrightarrow GL(4, \mathbf{Z}_2)\colon g_1 \longmapsto \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad g_2 \longmapsto \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix};$$

$\Delta_{23}\colon G_2 \longrightarrow \mathbf{Z}_3^\times\colon g \longmapsto 1;$

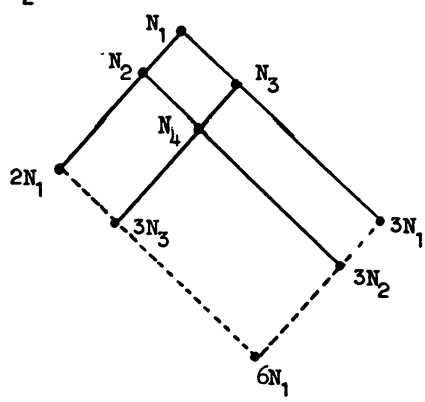
$$\Delta_{24}\colon G_2 \longrightarrow GL(4, \mathbf{Z}_3)\colon g_1 \longmapsto \begin{pmatrix} -1 & -1 & -1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad g_2 \longmapsto \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix}.$$

The lattices of centerings are

for G_1



for G_2



In analogy to Section 5 we denote by A_i, B_i the matrices whose columns are bases of the centerings M_i, N_i expressed in terms of the basis of M_1, N_1, respectively. Again they are ordered according to the corresponding forms.

Quadratic form: $F_1 = I_5$.

Basis of corresponding centering: $A_1 = I_5$.

Quadratic form: $F_2 = A_2 A_2^T$ with basis of corresponding centering

$$A_2 = (x, g_1x, g_1^2x, g_1^3x, y) \quad \text{and} \quad x^T = (1, 1, 0, 0, 0), \quad y^T = (0, 0, 0, 1, -1).$$

Quadratic form: $F_3 = A_3 A_3^T$ with basis of corresponding centering

$$A_3 = (x, g_1x, g_1^2x, g_1^3x, y) \quad \text{and} \quad x^T = (2, 0, 0, 0, 0), \quad y^T = (1, 1, 1, 1, 1).$$

Quadratic form: $F_4 = 6I_5 - J_5$.

Basis of corresponding centering: $B_1 = I_5$.

Quadratic form: $F_5 = I_5 + J_5$.

Basis of corresponding centering: $B_4 = I_5 + J_5$.

Quadratic form: $F_6 = B_2(6I_5 - J_5)B_2^T$ with basis of corresponding centering $B_2 = A_2$.

Quadratic form: $F_7 = B_3(6I_5 - J_5)B_3^T$ with basis of corresponding centering

$$B_3 = \begin{pmatrix} 2 & 2 & 2 & 2 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

An easy consideration shows $M_1^\# = M_1$, $M_2^\# = M_3$, $N_1^\# = N_4$, $N_2^\# = N_3$. Now some simple computations yield

(7.1) THEOREM. *There are exactly seven irreducible maximal finite subgroups of $GL(5, \mathbf{Z})$ up to $\mathbf{Z}$-equivalence. The automorphs of the quadratic forms $F_1, \ldots, F_7$ as given in this section form a full set of representatives. The automorphs of F_1, F_2, F_3 are rationally equivalent and isomorphic to the wreath product $C_2 \sim S_5$. Also, the automorphs of F_4, F_5, F_6, F_7 are rationally equivalent and isomorphic to $C_2 \times S_6$.*

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