Recurrence Formula of the Taylor Series Expansion Coefficients of the Jacobian Elliptic Functions

By Alois Schett

Abstract. A general recurrence formula permitting calculation of the Taylor series expansion coefficients of the Jacobian elliptic functions and the number of permutations of \( n \) natural numbers with a given run up or peak is given and its application is demonstrated.

In [1] we studied properties of the Taylor series expansion coefficients \( a_n \) comprising those of the Jacobian elliptic functions and tabulated them up to \( n = 15 \). Further tabulations of these coefficients for \( n = 16 \) to \( n = 50 \) are published in [2]. In the present paper we are giving a recurrence formula for the coefficients \( a_n \).

We recapitulate briefly for later use the properties of \( a_n \) studied in [1]:

1. \( a_n \) are triangle matrices with \((n_{\text{even}} + 2)/2\) or \((n_{\text{odd}} + 1)/2\) columns and rows.
2. \( a_n = A_n^T \).
3. The sum of the elements of \( A_n \) is equal to \( n! \).
4. The sum of the elements of a row \( i \) of \( A_n \) is the number of permutations of \( n \) natural numbers with \( i - 1 \) runs up.
5. The sum of the elements \( a_{i,j} \) of \( A_n \) with \( i + j \) constant \( > m \) (\( m \) maximal rows) is the number of permutations of \( n \) natural numbers with \( k = n - (i + j) - 1 \) peaks.
6. For \( n \) even and \( i + j = (n + 2)/2 + 1 \), \( (a_{i,j})_n = (a_{i,j})_{n+1} \). For \( n \) odd \( a_{i,(n+1)/2} = a_{i,(n+3)/2} \).
7. \( a_{(n+2)/2,(n+2)/2} = 0 \) for \( n \) even. \( a_{(n+1)/2,(n+1)/2} = 2^{n-1} \) for \( n \) odd.
8. The elements \( (a_{i,j})_n, i + j = (n + 2)/2 + 1 \), and \( (a_{i,(n+2)/2})_n \) \( (n \) even) are the Taylor series expansion coefficients of the Jacobian elliptic functions \( \text{sn}(u, k) \) and \( \text{cn}(u, k) \), \( \text{dn}(u, k) \), respectively.

The formal recurrence formula for \( A_n \) reads \( A_{n+1} = T_n A_n \). We have to find \( T_n \) and to define its application on \( A_n \). By means of mathematical induction we obtained the following results.

\( T_n \) are triangle matrices with the elements

\[ (t_{i,j;1}, t_{i,j;2}, t_{i,j;3})_n = (n - 2(j - 1), 3 + 2(j - 1) - n + 2i, n + 2 - 2i)_n, \]

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\[ (t_{i,j;1}, t_{i,j;2}, t_{i,j;3})_n = (0, 0, 0)_n \quad \text{for } i + j < n/2 + 1, \text{ } n \text{ even and} \]
\[ i + j < (n - 1)/2 + 1, \text{ } n \text{ odd,} \]
\[ i, j = 1, 2, 3, \ldots, n/2 \quad \text{for } n \text{ even and } (n - 1)/2 \text{ for } n \text{ odd.} \]

The symmetry
\[ (t_{i,j;1}, t_{i,j;2}, t_{i,j;3}) = (t_{j,i;3}, t_{j,i;2}, t_{j,i;1}) \]
and the relation
\[ t_{i,j;1} + t_{i,j;2} + t_{i,j;3} = n + 5 \]
are valid.

Applying \((t_{i,j;1}, t_{i,j;2}, t_{i,j;3})_n\) on \((a_{n,k})_n\) according to the relation
\[ (a_{i,j})_{n+1} = (a_{i,j-1} \cdot t_{i-1,j-1;1} + a_{i,j} \cdot t_{i-1,j-1;2} + a_{i-1,j} \cdot t_{i-1,j-1;3})_n \]
and using the properties 2. and 6. of \(A_n\) mentioned above permits calculation of the elements of \(A_{n+1}\).

Examples. We illustrate these formulas on \(n\) even and \(n\) odd.

\[ A_7 = T_6 A_6, \]

\[
A_6 = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 135 & 44 \\
0 & 135 & 328 & 16 \\
1 & 44 & 16 & 0
\end{pmatrix}, \quad T_6 = \begin{pmatrix}
0, 0, 0, 0, 0, 2, 3, 6 \\
0, 0, 4, 3, 4, 2, 5, 4 \\
6, 3, 2, 4, 5, 2, 7, 2
\end{pmatrix}
\]

\[ (a_{2,4})_7 = (a_{2,3} \cdot t_{1,3;1} + a_{2,4} \cdot t_{1,3;2} + a_{1,4} \cdot t_{1,3;3})_6 = 135 \cdot 2 + 44 \cdot 3 + 1 \cdot 6 = 408, \]
\[ (a_{3,4})_7 = (a_{3,3} \cdot t_{2,3;1} + a_{3,4} \cdot t_{2,3;2} + a_{2,4} \cdot t_{2,3;3})_6 = 328 \cdot 2 + 16 \cdot 5 + 44 \cdot 4 = 912, \]
\[ (a_{4,4})_7 = (a_{4,3} \cdot t_{3,3;1} + a_{4,4} \cdot t_{3,3;2} + a_{3,4} \cdot t_{3,3;3})_6 = 16 \cdot 2 + 0 \cdot 7 + 16 \cdot 2 = 64, \]
\[ (a_{3,3})_7 = (a_{3,2} \cdot t_{2,2;1} + a_{3,3} \cdot t_{2,2;2} + a_{2,3} \cdot t_{2,2;3})_6 = 135 \cdot 4 + 328 \cdot 3 + 135 \cdot 4 = 2064. \]

According to property 6 we obtain \(a_{1,4} = 1, a_{2,3} = 135\), and since \(A_n\) is symmetric all elements of \(A_7\) are known.

\[ A_8 = T_7 A_7, \]
\[
A_7 = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 135 & 408 \\
0 & 135 & 2064 & 912 \\
1 & 408 & 912 & 64
\end{pmatrix}, \quad T_7 = \begin{pmatrix}
0, 0, 0, 0, 0, 3, 2, 7 \\
0, 0, 5, 2, 5, 3, 4, 5 \\
7, 2, 3, 5, 4, 3, 6, 3
\end{pmatrix},
\]

\[
(a_{2,4})_8 = (a_{2,3} \cdot t_{1,3;1} + a_{2,4} \cdot t_{1,3;2} + a_{1,4} \cdot t_{1,3;3})_7 = 135 \cdot 3 + 408 \cdot 2 + 1 \cdot 7 = 1228,
\]

\[
(a_{3,4})_8 = (a_{3,3} \cdot t_{2,3;1} + a_{3,4} \cdot t_{2,3;2} + a_{2,4} \cdot t_{2,3;3})_7 = 2064 \cdot 3 + 912 \cdot 4 + 408 \cdot 5 = 11880,
\]

\[
(a_{4,4})_8 = (a_{4,3} \cdot t_{3,3;1} + a_{4,4} \cdot t_{3,3;2} + a_{3,4} \cdot t_{3,3;3})_7 = 912 \cdot 3 + 64 \cdot 6 + 912 \cdot 3 = 5856,
\]

\[
(a_{3,3})_8 = (a_{3,2} \cdot t_{2,2;1} + a_{3,3} \cdot t_{2,2;2} + a_{2,3} \cdot t_{2,2;3})_7 = 135 \cdot 5 + 2064 \cdot 2 + 135 \cdot 5 = 5478.
\]

Using the properties 6. and 2. of \( A_n \), we obtain the remaining elements of \( A_8 \).

\[
A_8 = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1228 & 408 \\
0 & 0 & 5478 & 11880 & 912 \\
0 & 1228 & 11880 & 5856 & 64 \\
1 & 408 & 912 & 64 & 0
\end{pmatrix}.
\]

In conclusion, the present recurrence formula permits calculation of the Taylor series expansion coefficients of the Jacobian elliptic functions and the number of permutations of \( n \) natural numbers with a given run up or peak.

Centre d'Etudes Nucléaires
Saclay, France
