Roots of Two Transcendental Equations
Determining the Frequency Spectra of Standing Spherical Electromagnetic Waves

By Robert L. Pexton and Arno D. Steiger

Abstract. Roots of the transcendental equations

\[
\frac{j_l(\alpha \sqrt{|\epsilon|})}{y_l(\alpha \sqrt{|\epsilon|})} = \frac{1 - j_l(\alpha \sqrt{|\epsilon|})}{1 + j_l(\alpha \sqrt{|\epsilon|})}
\]

and

\[
\frac{\eta j_{l-1}(\eta)}{\eta y_{l-1}(\eta)} = \frac{1 + |\epsilon|^\lambda}{1 - |\epsilon|^\lambda}
\]

for the spherical Bessel functions of the first and second kind, \(j_l(x)\) and \(y_l(x)\), and for the modified spherical Bessel functions of the first kind, \(i_l(x)\), have been computed. The ranges for the parameters \(\sqrt{|\epsilon|}\) and \(\alpha\), the order \(l\) and the root index \(n\) are:

\[
\sqrt{|\epsilon|} = 1.0, 10.0, 100.0, 500.0; \quad \alpha = 0.1(0.1)0.7; \quad l = 1(1)15; \quad n = 1(1)30.
\]

In a previous communication [1] roots of two transcendental equations involving spherical Bessel functions were presented. These roots correspond to the eigen-frequency spectra of the transverse electric and the transverse magnetic multipole fields in the domain bounded by two perfectly conducting concentric spheres \(r = \)
STANDING SPHERICAL ELECTROMAGNETIC WAVES

1001

aR and r = R, 0 < a < 1). On the surface of a perfect conductor the tangential components of the electric field and the normal component of the magnetic field vanish. In the present work these boundary conditions are assumed to hold only on the outer sphere. The inner sphere is regarded as the boundary of a nontransparent plasma core into which the electromagnetic fields penetrate as evanescent (spherical) waves.

The necessary and sufficient conditions to ensure continuity of the electromagnetic field are that the tangential components of the electric and magnetic field vectors be continuous across this boundary surface, which is assumed to be charge-free and current-free.

From these boundary conditions then follow the characteristic equations [2]

\[
\begin{align*}
\frac{j_l(\lambda)}{y_l(\lambda)} &= \frac{\sqrt{\epsilon} j_{i-1}(\alpha \lambda \sqrt{\epsilon}) j_l(\alpha \lambda) - j_{l-1}(\alpha \lambda) j_l(\alpha \lambda \sqrt{\epsilon})}{\sqrt{\epsilon} j_{i-1}(\alpha \lambda \sqrt{\epsilon}) y_l(\alpha \lambda) - y_{l-1}(\alpha \lambda) j_l(\alpha \lambda \sqrt{\epsilon})}, \\
\eta j_{i-1}(\eta) - l_i(\eta) &= \frac{\alpha \eta \sqrt{\epsilon} [j_{i-1}(\alpha \eta \sqrt{\epsilon}) j_i(\alpha \eta) - \sqrt{\epsilon} j_{i-1}(\alpha \eta) j_i(\alpha \eta \sqrt{\epsilon})] + (\epsilon - 1) j_i(\alpha \eta \sqrt{\epsilon}) j_l(\alpha \eta)}{\alpha \eta \sqrt{\epsilon} [j_{i-1}(\alpha \eta \sqrt{\epsilon}) y_i(\alpha \eta) - \sqrt{\epsilon} y_{i-1}(\alpha \eta) j_i(\alpha \eta \sqrt{\epsilon})] + (\epsilon - 1) j_i(\alpha \eta \sqrt{\epsilon}) y_l(\alpha \eta)}.
\end{align*}
\]

The spherical Bessel functions of the first and second kind, \(j_l(z)\) and \(y_l(z)\), are defined in [3, p. 437]. The nth root, \(\lambda_{l,n}\), of Eq. (1) is proportional to the nth characteristic frequency of the transverse electric \(2l\)-pole field. The nth root, \(\eta_{l,n}\), of Eq. (2) is proportional to the nth characteristic frequency of the transverse magnetic \(2l\)-pole field. The dielectric constant \(\epsilon\) is in general a complex number. If \(\epsilon\) is real and negative (\(\epsilon = -|\epsilon|\)), the electromagnetic field enters the plasma core as an evanescent wave without being absorbed (total internal reflection). With this choice of \(\epsilon\) and the relations [3, pp. 443, 469]

\[
\begin{align*}
j_l(iz) &= e^{il\pi/2} \sqrt{\frac{n}{2z}} I_{l+\frac{1}{2}}(z), \quad (-\pi < \arg z \leq \frac{\pi}{2}), \\
j_l(x) &= \sqrt{\frac{\pi}{2x}} I_{l+\frac{1}{2}}(x),
\end{align*}
\]

where \(I_{l+\frac{1}{2}}(x)\) is the modified Bessel function of the first kind, Eqs. (1) and (2) become

\[
\begin{align*}
\frac{j_l(\lambda)}{y_l(\lambda)} &= \frac{j_l(\alpha \lambda \sqrt{|\epsilon|}) i_{l-1}(\alpha \lambda \sqrt{|\epsilon|}) - i_{l-1}(\alpha \lambda) j_l(\alpha \lambda \sqrt{|\epsilon|})}{\sqrt{|\epsilon|} i_{l-1}(\alpha \lambda) y_l(\alpha \lambda) - y_{l-1}(\alpha \lambda) i_l(\alpha \lambda \sqrt{|\epsilon|})}, \\
\eta j_{i-1}(\eta) - l_i(\eta) &= \frac{\alpha \eta \sqrt{\epsilon} [j_{i-1}(\alpha \eta \sqrt{\epsilon}) i_i(\alpha \eta) - \sqrt{\epsilon} i_{i-1}(\alpha \eta) j_i(\alpha \eta \sqrt{\epsilon})] + (\epsilon - 1) j_i(\alpha \eta \sqrt{\epsilon}) i_l(\alpha \eta)}{\alpha \eta \sqrt{\epsilon} [j_{i-1}(\alpha \eta \sqrt{\epsilon}) y_i(\alpha \eta) - \sqrt{\epsilon} y_{i-1}(\alpha \eta) i_i(\alpha \eta \sqrt{\epsilon})] + (\epsilon - 1) j_i(\alpha \eta \sqrt{\epsilon}) y_l(\alpha \eta)}.
\end{align*}
\]

and
\[
\frac{\eta j_{l-1}(\eta) - l j_l(\eta)}{\eta y_{l-1}(\eta) - l y_l(\eta)}
\]

In the limit \( |\epsilon| \to \infty \) the equations (3) and (4) become, respectively, identical with the equations (1) and (2') of reference [1].

The calculation of the roots of Eqs. (3) and (4) was performed as follows. First, the equations were each expressed in the form \( F(u) = 0 \). Then, for a given order \( l \), a fixed parameter \( \alpha \), and a given value for the absolute magnitude of the dielectric constant \( \epsilon \), the function \( F(u) \) was evaluated at a sequence of points \( u_j \), for which \( u_j - u_{j-1} = \) constant, until a sign change occurred. In this last interval a root was computed by using a modified Müller technique [4]. The spherical Bessel functions were computed by means of Meichel’s recurrence method [5]. The ratios of the modified spherical Bessel functions were computed using Lentz’s continued fraction technique [6].

The numerical values for the roots of Eqs. (3) and (4) which are listed in the microfiche supplement of this issue are accurate to at least 10 significant figures. The ranges for the square root of the absolute magnitude of the dielectric constant \( \epsilon \), the parameter \( \alpha \), the order \( l \) and the root index \( n \) are

\[
\sqrt{|\epsilon|} = 1.0, 10.0, 100.0, 500.0;
\]

\[
\alpha = 0.1(0.1)0.7;
\]

\[
l = 1(1)15;
\]

\[
n = 1(1)30.
\]

These calculations were performed on a CDC 7600 computer.

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