An Approximation for $\int_x^\infty e^{-t^2/2} t^p \, dt$, $x > 0$, $p$ real

By A. R. DiDonato

Abstract. A new approximation is given for $\int_x^\infty e^{-t^2/2} t^p \, dt$, $x > 0$, $p$ real, which extends an earlier approximation of Boyd's for $p = 0$.


$$g(x) = 4/[3x + \sqrt{x^2 + 8}]$$

as an approximation for

$$F(x) \equiv [1/Z(x)] \int_x^\infty Z(t) \, dt, \quad x > 0,$$

where

$$Z(x) = \exp(-x^2/2).$$

It can be shown $g(x) > F(x)$, and in fact

$$g(x) - F(x) = 2x^{-7} + O(x^{-9}), \quad (x \to \infty).$$

For $x > c \equiv (4 - \pi)/\sqrt{\pi(\pi - 2)} \approx 0.453$, $g(x)$ serves as a much better approximation to $F(x)$ than the well-known estimate

$$h(x) = 2/[x + \sqrt{x^2 + 8/\pi}], \quad [1, \text{p. 298}].$$

More specifically, it is easy to conclude that

$$F(x) < g(x) < h(x), \quad x > c,$$

as Table I below indicates.

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Our objective is to generalize Boyd's result to the function

$$F(p, x) \equiv Y(p, x)/Z(p, x),$$

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where

\[ Y(p, x) \equiv \int_x^\infty Z(p, t) \, dt, \]

\[ Z(p, x) = x^p \exp(-x^2/2), \quad x > 0, \quad p \text{ real}. \]

The corresponding approximation for \( F(p, x) \) is given by

\[ g(p, x) = 4x/[3(x^2 - p) + \sqrt{(x^2 - p)^2 + 8(x^2 + p)}], \]

which reduces to (1) for \( p = 0 \), and also is exact for \( p = 1 \), i.e., \( g(1, x) = F(1, x) \).

For fixed \( p \), it improves as \( x \) increases and, depending on the value of \( p \), it bounds \( F \) either from above or below for all \( x > x_m \). In particular,

\[ \begin{cases} (a) & p < 0, \quad x^2 \geq x_m^2 = -p \quad (\Rightarrow g(p, x) > F(p, x)), \\ (b) & 0 < p \leq 1, \quad x^2 \geq x_m^2 = p + 2p^2/3 \quad (\Rightarrow g(p, x) \geq F(p, x))^*, \\ (c) & p > 1, \quad x^2 \geq x_m^2 = p + 2p^2/3 \quad (\Rightarrow g(p, x) < F(p, x)). \end{cases} \]

By expanding (10) in powers of \( 1/x \) and subsequently taking the difference of the leading terms with those of the asymptotic series for \( F(p, x) \),

\[ F(p, x) \equiv \frac{1}{x} \left[ 1 + \frac{p - 1}{x^2} + \frac{(p - 1)(p - 3)}{x^4} + \cdots \right], \quad (x \to \infty), \]

we find

\[ g(p, x) - F(p, x) \approx \frac{2(1 - p)}{x^7} + \frac{2(1 - p)(4p - 19)}{x^9} + O(x^{-11}), \quad (x \to \infty). \]

Table II is given to show the comparison between \( F(p, x) \) and \( g(p, x) \) for some selected values of \( p \) and \( x \). The asterisked \( x \) values are close approximates of \( x_m \) given in (11).

Before deriving (10), we note that an approximation, \( g_1(p, x) \), for \( F \) can also be obtained from the first two terms of the continued fraction expansion for the incomplete gamma function, namely

\[ g_1(p, x) = \frac{x^2 + 2}{x(x^2 + 3 - p)}, \quad [4, p. 356]. \]

The relationship between \( F \) and the incomplete gamma function is given below in (29).

From (12) and (14) one obtains

\[ g_1(p, x) - F(p, x) \approx 2(p - 1)(p - 3)/x^7 + O(x^{-9}), \quad (x \to \infty). \]

A comparison with (13) shows that at large \( x \), \( g_1 \) is better than \( g \) for \( 2 < p < 4 \), but it is not as good otherwise, especially at large \( |p| \).

*For \( 0 < p < 1 \), computer results indicate \( 3p \) as a sharper estimate for \( x_m^2 \).

**Since the algebra is somewhat lengthy, the first two terms on the right-hand side of (12) and (13) were also computed by A. Morris, as well as three additional ones, using his algebraic computer program, "FLAP," [3].
**Table II** (see Table I for \( p = 0 \))

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Derivation of (10) and (11). We now derive (10) treating (11(a), (b) and (c) separately. The final results depend on the function \( H(p, x) \) given below by (19) or (20).

Let

\[
(16) \quad f(p, x) = \frac{1}{F(p, x)} = \frac{Z(p, x)}{Y(p, x)}.
\]

For brevity, denote \( f(p, x) \) by \( f \), \( \partial f/\partial x \) by \( f' \), \( \partial^2 f/\partial x^2 \) by \( f'' \). Then

\[
(17) \quad f' = f(f + p/x - x) = fu, \quad u = f + p/x - x,
\]

\[
(18) \quad f'' = fH, \quad H = H(p, x),
\]

where

\[
(19) \quad H = 2f^2 + 3(p/x - x)f + (p/x - x)^2 - 1 - p/x^2,
\]

or

\[
(20) \quad H = 2u^2 + (x - p/x)u - 1 - p/x^2 = u' + u^2.
\]

We shall use the following properties of \( H \), which are easily found:
\[ \lim_{x \to 0^+} H(p, x) = \begin{cases} \infty, & p < 0, \\ \frac{(4 - \pi)}{\pi}, & p = 0, \\ -\infty, & 0 < p < 1, \\ \infty, & p > 1. \end{cases} \]

Also
\[ H(p, x) \equiv 2(1 - p)/x^4, \quad (x \to \infty), \]

which follows after some tedious algebra from (12) and (19).

For \( p < 0 \), we have
\[ u = f + p/x - x > 0, \quad x > s = \sqrt{-p}. \]

Indeed, let \( S = uY = Z - (x - p/x)Y \), so that \( \partial S/\partial x \leq 0 \) for \( x^2 \geq -p \), and \( S \equiv Z/x^2 \) \( (x \to \infty) \). Hence \( S > 0 \) for \( x^2 \geq -p \), and since \( Y \) is always positive (23) follows.

From this result, with (21) and (22), we have \( H > 0 \) for \( x \in [s, \infty) \). In fact, if this were not the case, there would exist a point \( \xi \in (s, \infty) \) for which \( H(p, \xi) = 0 \), \( H'(p, \xi) > 0 \). But this is impossible since
\[ H'(p, x) = (f + 2u)H(p, x) - 2u^3 + 2p/x^3 \]
is negative if \( H = 0 \), \( u > 0 \), and \( p < 0 \).

Thus factoring \( H \), as given in (19),
\[ H = (f - \eta_+)(f - \eta_-) > 0, \quad x^2 \geq -p, \quad p < 0, \]
we get (10) with 11(a) from \( f - \eta_+ > 0 \), where
\[ \eta_\pm(p, x) = [3(x - p/x) \pm \sqrt{(x - p/x)^2 + 8(1 + p/x^2)}]/4. \]

Now consider the case \( p > 1 \). From (21) and (22) we know \( H \) has at least one positive zero such that \( H'(p, x_0) \leq 0 \), where \( x_0 \) denotes the largest such zero. Moreover, if \( z \) denotes the largest zero of \( H \) with \( H'(p, z) > 0 \), then \( z < x_0 \). Thus \( H \leq 0 \) for all \( x > x_0 \). In order to get an estimate, \( x_m \), of \( x_0 \) we have from (24) and \( H'(p, x_0) \leq 0 \), that
\[ u(p, x_0) \geq p^{1/3}/x_0, \quad p > 1. \]

Inequality 11(c) now follows directly by using (27) and \( f(p, x_0) = \eta_+(p, x_0) \) in the expression for \( u \) given in (17).

When \( 0 < p < 1 \), the analysis used to obtain 11(b) is similar to that for \( p > 1 \). First, it is shown (27) holds with the inequality reversed. Then proceeding as above, one obtains 11(b) with \( H(p, x) > 0 \) for all \( x > x_m > x_0 \). The details are omitted.

**Relation of \( F(p, x) \) to the Incomplete Gamma Function.** The quantity \( F(p, x) \) can be related to the normalized incomplete gamma function. Let
\[ r = t^2/2, \quad y = x^2/2, \]
so that
AN APPROXIMATION

\[ F(p, x) = \frac{1}{\sqrt{2}} e^{y} y^{-p/2} \Gamma \left( \frac{p + 1}{2}, y \right), \]

where

\[ \Gamma(a, z) = \int_{z}^{\infty} e^{-r} r^{a-1} \, dr, \quad z > 0, \ a \ \text{real}. \]

Thus, we have from (10) and (11)

\[ \Gamma(a, y) = \left( \frac{4e^{-y} y^a}{3[y - a + \frac{1}{2}] + \sqrt{(y - a + \frac{1}{2})^2 + 4(y + a - \frac{1}{2})}} \right), \]

with

\[ \begin{cases} 
(a) \ a \leq \frac{1}{2}, \quad y \geq y_m \equiv \frac{1}{2} - a, & (\Rightarrow g(2a - 1, \sqrt{2}y) > \Gamma(a, y)), \\
(b) \ \frac{1}{2} < a < 1, \quad y \geq y_m \equiv a - \frac{1}{2} + (2a - 1)^{2/3}, & (\Rightarrow g(2a - 1, \sqrt{2}y) \geq \Gamma(a, y)), \\
(c) \ a \geq 1, \quad y \geq y_m \equiv a - \frac{1}{2} + (2a - 1)^{2/3}, & (\Rightarrow g(2a - 1, \sqrt{2}y) \leq \Gamma(a, y)).
\end{cases} \]

An Application. The function \( g(p, x) \) is particularly good for giving quick estimates when \( p < 0 \) and \( F(p, x) \) cannot be evaluated by the usual recurrence relationship on \( p \). In addition, \( g \) can often be used to establish properties of \( F \). For example, a result not obtained easily by direct means, is to find, for \( p < 0 \), a good estimate to \( x = X \), where \( F(p, X) > F(p, x) \) for all \( x > 0 \). In fact, by using (10) and noting that \( F'(p, X) = 0 \) requires \( F(p, X) = X/(X^2 - p) \), we find

\[ g(p, X) - F(p, X) \approx -X(X^2 + p)/(X^2 - p)^3. \]

Therefore, with \( X^2 \approx -p \),

\[ F(p, X) \approx F(p, \sqrt{-p}) \approx g(p, \sqrt{-p}) = 1/(2\sqrt{-p}). \]

Clearly (34) also implies that \( F(p, X) \leq 1/(2\sqrt{-p}) \). Thus, if \( p = -100 \), then \( X = 9.950374, \sqrt{-p} = 10, F(-100, X) = .04999938, 1/(2\sqrt{-p}) = .05, g(p, X) = .05000063 \). Even for the case of \( p = -4 \), where one does not expect the right side of (33) to hold, we find \( X = 1.791507, \sqrt{-p} = 2, F(-4, X) = .2484926, 1/(2\sqrt{-p}) = .250, g(-4, X) = .2524567. \)

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