MATHEMATICS OF COMPUTATION, VOLUME 32, NUMBER 141
JANUARY 1978, PAGES 271–275

# An Approximation for $\int_x^\infty e^{-t^2/2} t^p \, dt$, $x > 0$, $p$ real

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**Abstract.** A new approximation is given for $\int_x^\infty e^{-t^2/2} t^p \, dt$, $x > 0$, $p$ real, which extends an earlier approximation of Boyd's for $p = 0$.

**Introduction.** In [2] Boyd gives

$$g(x) = 4/[3x + \sqrt{x^2 + 8}] \tag{1}$$

as an approximation for

$$F(x) \equiv [1/Z(x)] \int_x^\infty Z(t)\, dt, \qquad x > 0, \tag{2}$$

where

$$Z(x) = \exp(-x^2/2). \tag{3}$$

It can be shown $g(x) > F(x)$, and in fact

$$g(x) - F(x) = 2x^{-7} + O(x^{-9}), \qquad (x \to \infty). \tag{4}$$

For $x > c \equiv (4 - \pi)/\sqrt{\pi(\pi - 2)} \cong .453$, $g(x)$ serves as a much better approximation to $F(x)$ than the well-known estimate

$$h(x) = 2/[x + \sqrt{x^2 + 8/\pi}], \qquad [1, \text{p. } 298]. \tag{5}$$

More specifically, it is easy to conclude that

$$F(x) < g(x) < h(x), \qquad x > c, \tag{6}$$

as Table I below indicates.

TABLE I

| $x$ | $F(x)$ | $g(x)$ | $h(x)$ |
|---|---|---|---|
| 0.5 | .8763645 | .9148542 | .9206969 |
| 1.0 | .6556795 | .6666667 | .6936713 |
| 2.0 | .4213692 | .4226497 | .4387303 |
| 5.0 | .1928081 | .1928216 | .1951510 |
| 9.5 | .1041337 | .1041338 | .1045309 |

Our objective is to generalize Boyd's result to the function

$$F(p, x) \equiv Y(p, x)/Z(p, x), \tag{7}$$

Received April 11, 1977.
*AMS (MOS) subject classifications* (1970). Primary 41A30; Secondary 65D20.

where

$$Y(p, x) \equiv \int_x^\infty Z(p, t)\, dt, \tag{8}$$

$$Z(p, x) \equiv x^p \exp(-x^2/2), \qquad x > 0, \qquad p \text{ real}. \tag{9}$$

The corresponding approximation for $F(p, x)$ is given by

$$g(p, x) = 4x/[3(x^2 - p) + \sqrt{(x^2 - p)^2 + 8(x^2 + p)}]\,, \tag{10}$$

which reduces to (1) for $p = 0$, and also is exact for $p = 1$, i.e., $g(1, x) = F(1, x)$. For fixed $p$, it improves as $x$ increases and, depending on the value of $p$, it bounds $F$ either from above or below for all $x > x_m$. In particular,

$$\begin{cases} \text{(a)} \quad p \leqslant 0, & x^2 \geqslant x_m^2 \equiv -p & (\Rightarrow g(p, x) > F(p, x)), \\ \text{(b)} \quad 0 < p \leqslant 1, & x^2 \geqslant x_m^2 \equiv p + 2p^{2/3} & (\Rightarrow g(p, x) \geqslant F(p, x))^*, \\ \text{(c)} \quad p \geqslant 1, & x^2 \geqslant x_m^2 \equiv p + 2p^{2/3} & (\Rightarrow g(p, x) \leqslant F(p, x)). \end{cases} \tag{11}$$

By expanding (10) in powers of $1/x$ and subsequently taking the difference of the leading terms with those of the asymptotic series for $F(p, x)$,

$$F(p, x) \cong \frac{1}{x}\left[1 + \frac{p-1}{x^2} + \frac{(p-1)(p-3)}{x^4} + \cdots\right], \qquad (x \to \infty), \tag{12}$$

we find

$$g(p, x) - F(p, x) \cong \frac{2(1-p)}{x^7} + \frac{2(1-p)(4p-19)}{x^9} + O(x^{-11}), \qquad (x \to \infty).^{**} \tag{13}$$

Table II is given to show the comparison between $F(p, x)$ and $g(p, x)$ for some selected values of $p$ and $x$. The asterisked $x$ values are close approximates of $x_m$ given in (11).

Before deriving (10), we note that an approximation, $g_1(p, x)$, for $F$ can also be obtained from the first two terms of the continued fraction expansion for the incomplete gamma function, namely

$$g_1(p, x) = \frac{x^2 + 2}{x(x^2 + 3 - p)}, \qquad [4, \text{p. } 356]. \tag{14}$$

The relationship between $F$ and the incomplete gamma function is given below in (29).

From (12) and (14) one obtains

$$g_1(p, x) - F(p, x) \cong 2(p-1)(p-3)/x^7 + O(x^{-9}), \qquad (x \to \infty). \tag{15}$$

A comparison with (13) shows that at large $x$, $g_1$ is better than $g$ for $2 < p < 4$, but it is not as good otherwise, especially at large $|p|$.

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*For $0 < p < 1$, computer results indicate $3p$ as a sharper estimate for $x_m^2$.

**Since the algebra is somewhat lengthy, the first two terms on the right-hand side of (12) and (13) were also computed by A. Morris, as well as three additional ones, using his algebraic computer program, "FLAP," [3].

TABLE II (see Table I for $p = 0$)

| p | x | F(p,x) | g(p,x) | p | x | F(p,x) | g(p,x) |
|---|---|---|---|---|---|---|---|
| -100 | 1 | .0099980 | .0100000 | -40 | 1 | .0249672 | .0250000 |
| | 5 | .0401906 | .0401948 | | 6.3* | .0790482 | .0790602 |
| | 10* | .0499988 | .0500000 | | 8 | .0767483 | .0767535 |
| | 13 | .0482807 | .0482812 | | 12 | .0650181 | .0650189 |
| -10 | .75 | .0772836 | .0795903 | -4 | 1 | .2185598 | .2500000 |
| | 3* | .1579338 | .1583358 | | 2* | .2473026 | .2500000 |
| | 8 | .1070674 | .1070723 | | 5 | .1683551 | .1684001 |
| | 12 | .0774887 | .0774891 | | 10 | .0953216 | .0953223 |
| -1 | 1* | .4614553 | .5000000 | -.8 | .90* | .5032524 | .5568742 |
| | 2 | .3613286 | .3636364 | | 2 | .3721367 | .3742823 |
| | 5 | .1860899 | .1861126 | | 5 | .1873965 | .1874189 |
| | 10 | .0980755 | .0980759 | | 10 | .0982647 | .0982650 |
| -.4 | .64* | .6399281 | .7794144 | .2 | .5 | 1.018194 | .9765504 |
| | .80 | .6074871 | .6594522 | | .77* | .8242077 | .8262394 |
| | 1 | .5652979 | .5882353 | | 1 | .7091266 | .7142857 |
| | 5 | .1900660 | .1900842 | | 3 | .3099818 | .3101590 |
| | 10 | .0986452 | .0986454 | | 7 | .1406465 | .1406479 |
| .6 | .8 | .9957427 | .9786360 | 1.4 | 1.4 | .8090514 | .8122124 |
| | 1.35* | .6549141 | .6556292 | | 1.98* | .5444214 | .5441366 |
| | 3 | .3212850 | .3213781 | | 4 | .2557314 | .2557085 |
| | 7 | .1417435 | .1417442 | | 8 | .1257627 | .1257624 |
| 3 | 2 | .7500000 | .7583057 | 9 | 3 | .9344615 | 1.000000 |
| | 2.68* | .4770367 | .4765579 | | 4.2* | .3920753 | .3917566 |
| | 4 | .2812500 | .2811137 | | 7 | .1692794 | .1692546 |
| | 8 | .1289062 | .1289046 | | 10 | .1084996 | .1084977 |
| 19 | 5 | .5265147 | .5305361 | 39 | 7 | .4765338 | .4798227 |
| | 5.77* | .3404467 | .3402561 | | 7.87* | .2989724 | .2988727 |
| | 8 | .1712925 | .1712553 | | 11 | .1311763 | .1311612 |
| | 10 | .1213368 | .1213307 | | 20 | .0552170 | .0552169 |
| 99 | 11 | .3867447 | .3881067 | 199 | 15 | .4202082 | .4231882 |
| | 11.9* | .2519044 | .2518658 | | 16.3* | .2245087 | .2244941 |
| | 15 | .1167631 | .1167549 | | 20 | .0980926 | .0980889 |
| | 20 | .0660856 | .0660851 | | 30 | .0427010 | .0427009 |

**Derivation of (10) and (11).** We now derive (10) treating 11(a), (b) and (c) separately. The final results depend on the function $H(p, x)$ given below by (19) or (20).

Let

$$f(p, x) \equiv 1/F(p, x) = Z(p, x)/Y(p, x). \tag{16}$$

For brevity, denote $f(p, x)$ by $f$, $\partial f/\partial x$ by $f'$, $\partial^2 f/\partial x^2$ by $f''$. Then

$$f' = f(f + p/x - x) = fu, \qquad u \equiv f + p/x - x, \tag{17}$$

$$f'' = fH, \qquad H = H(p, x), \tag{18}$$

where

$$H = 2f^2 + 3(p/x - x)f + (p/x - x)^2 - 1 - p/x^2, \tag{19}$$

or

$$H = 2u^2 + (x - p/x)u - 1 - p/x^2 = u' + u^2. \tag{20}$$

We shall use the following properties of $H$, which are easily found:

$$\lim_{x\to 0_+} H(p, x) = \begin{cases} \infty, & p < 0, \\ (4-\pi)/\pi, & p = 0, \\ -\infty, & 0 < p < 1, \\ \infty, & p > 1. \end{cases} \tag{21}$$

Also

$$H(p, x) \cong 2(1-p)/x^4, \quad (x \longrightarrow \infty), \tag{22}$$

which follows after some tedious algebra from (12) and (19).

For $p < 0$, we have

$$u = f + p/x - x > 0, \quad x \geqslant s \equiv \sqrt{-p}. \tag{23}$$

Indeed, let $S \equiv uY = Z - (x - p/x)Y$, so that $\partial S/\partial x \leqslant 0$ for $x^2 \geqslant -p$, and $S \cong Z/x^2$ $(x \longrightarrow \infty)$. Hence $S > 0$ for $x^2 \geqslant -p$, and since $Y$ is always positive (23) follows.

From this result, with (21) and (22), we have $H > 0$ for $x \in [s, \infty)$. In fact, if this were not the case, there would exist a point $\zeta \in (s, \infty)$ for which $H(p, \zeta) = 0$, $H'(p, \zeta) \geqslant 0$. But this is impossible since

$$H'(p, x) = (f + 2u)H(p, x) - 2u^3 + 2p/x^3 \tag{24}$$

is negative if $H = 0$, $u > 0$, and $p \leqslant 0$.

Thus factoring $H$, as given in (19),

$$H = (f - \eta_+)(f - \eta_-) > 0, \quad x^2 \geqslant -p, p < 0, \tag{25}$$

we get (10) with 11(a) from $f - \eta_+ > 0$, where

$$\eta_\pm(p, x) = [3(x - p/x) \pm \sqrt{(x - p/x)^2 + 8(1 + p/x^2)}]/4. \tag{26}$$

Now consider the case $p > 1$. From (21) and (22) we know $H$ has at least one positive zero such that $H'(p, x_0) \leqslant 0$, where $x_0$ denotes the largest such zero. Moreover, if $z$ denotes the largest zero of $H$ with $H'(p, z) > 0$, then $z < x_0$. Thus $H \leqslant 0$ for all $x \geqslant x_0$. In order to get an estimate, $x_m$, of $x_0$ we have from (24) and $H'(p, x_0) \leqslant 0$, that

$$u(p, x_0) \geqslant p^{1/3}/x_0, \quad p \geqslant 1. \tag{27}$$

Inequality 11(c) now follows directly by using (27) and $f(p, x_0) = \eta_+(p, x_0)$ in the expression for $u$ given in (17).

When $0 < p < 1$, the analysis used to obtain 11(b) is similar to that for $p > 1$. First, it is shown (27) holds with the inequality reversed. Then proceeding as above, one obtains 11(b) with $H(p, x) > 0$ for all $x > x_m > x_0$. The details are omitted.

**Relation of $F(p, x)$ to the Incomplete Gamma Function.** The quantity $F(p, x)$ can be related to the normalized incomplete gamma function. Let

$$r = t^2/2, \quad y = x^2/2, \tag{28}$$

so that

$$F(p, x) = \frac{1}{\sqrt{2}} e^{y} y^{-p/2} \Gamma\left(\frac{p+1}{2}, y\right), \tag{29}$$

where

$$\Gamma(a, z) \equiv \int_z^\infty e^{-r} r^{a-1}\, dr, \qquad z > 0,\ a \text{ real}. \tag{30}$$

Thus, we have from (10) and (11)

$$\Gamma(a, y) = \left(\frac{4e^{-y} y^{a}}{3[y - a + \tfrac{1}{2}] + \sqrt{(y - a + \tfrac{1}{2})^2 + 4(y + a - \tfrac{1}{2})}}\right), \tag{31}$$

with

$$\begin{cases} \text{(a) } a \leqslant \tfrac{1}{2}, & y \geqslant y_m \equiv \tfrac{1}{2} - a, & (\Rightarrow g(2a-1, \sqrt{2y}) > \Gamma(a, y)), \\ \text{(b) } \tfrac{1}{2} < a \leqslant 1, & y \geqslant y_m \equiv a - \tfrac{1}{2} + (2a-1)^{2/3}, & (\Rightarrow g(2a-1, \sqrt{2y}) \geqslant \Gamma(a, y)), \\ \text{(c) } a \geqslant 1, & y \geqslant y_m \equiv a - \tfrac{1}{2} + (2a-1)^{2/3}, & (\Rightarrow g(2a-1, \sqrt{2y}) \leqslant \Gamma(a, y)). \end{cases} \tag{32}$$

**An Application.** The function $g(p, x)$ is particularly good for giving quick estimates when $p < 0$ and $F(p, x)$ cannot be evaluated by the usual recurrence relationship on $p$. In addition, $g$ can often be used to establish properties of $F$. For example, a result not obtained easily by direct means, is to find, for $p < 0$, a good estimate to $x = X$, where $F(p, X) \geqslant F(p, x)$ for all $x > 0$. In fact, by using (10) and noting that $F'(p, X) = 0$ requires $F(p, X) = X/(X^2 - p)$, we find

$$g(p, X) - F(p, X) \cong -X(X^2 + p)/(X^2 - p)^3. \tag{33}$$

Therefore, with $X^2 \cong -p$,

$$F(p, X) \cong F(p, \sqrt{-p}) \cong g(p, \sqrt{-p}) = 1/(2\sqrt{-p}). \tag{34}$$

Clearly (34) also implies that $F(p, X) \leqslant 1/(2\sqrt{-p})$. Thus, if $p = -100$, then $X = 9.950374$, $\sqrt{-p} = 10$, $F(-100, X) = .04999938$, $1/(2\sqrt{-p}) = .05$, $g(p, X) = .05000063$. Even for the case of $p = -4$, where one does not expect the right side of (33) to hold, we find $X = 1.791507$, $\sqrt{-p} = 2$, $F(-4, X) = .2484926$, $1/(2\sqrt{-p}) = .250$, $g(-4, X) = .2524567$.

**Acknowledgment.** My thanks to Mr. Richard K. Hageman for his programming help, and the computation of Tables I and II.

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