Positivity of the Weights of Extended Gauss-Legendre Quadrature Rules

By Giovanni Monegato

Abstract. We show that the weights of extended Gauss-Legendre quadrature rules are all positive.

1. Introduction. We consider extended Gauss-Legendre quadrature formulas, i.e., integration rules of the type

\[ \int_{-1}^{1} f(x) dx = \sum_{i=1}^{n} A_i^{(n)} f(\xi_i^{(n)}) + \sum_{j=1}^{n+1} B_j^{(n)} f(x_j^{(n)}) + R_n(f), \]

where \( \xi_i^{(n)}, i = 1, \ldots, n \), are the zeros of the \( n \)th degree Legendre polynomial \( P_n(x) \), while the nodes \( x_j^{(n)}, j = 1, 2, \ldots, n + 1 \), and the weights \( A_i^{(n)}, B_j^{(n)} \) are chosen so that (1) has degree of exactness \( p = 3n + 1 \) (\( 3n + 2 \) if \( n \) is odd), i.e., \( R_n(f) = 0 \) whenever \( f \) is a polynomial of degree up to \( p \). If we denote by \( E_{n+1}(x) \) the polynomial of degree \( n + 1 \), whose zeros are the abscissas \( x_j^{(n)}, j = 1, 2, \ldots, n + 1 \), then \( E_{n+1}(x) \) has to satisfy the following orthogonality relation

\[ \int_{-1}^{1} P_n(x) E_{n+1}(x) x^k dx = 0, \quad k = 0, 1, \ldots, n. \]

Szegö [4] has studied \( E_{n+1}(x) \) in a different context and gives some very interesting results. For instance, he proves that the nodes \( x_j^{(n)} \) are in \((-1, 1)\) and interlace with the zeros of \( P_n(x) \).

Formulas for the computation of the weights \( A_i^{(n)} \) and \( B_j^{(n)} \) are given in [2], [3]. In [2] it is shown that the \( B_j^{(n)} \)'s are positive; however, nothing has been said about the sign of \( A_i^{(n)} \). In this note we show that the weights \( A_i^{(n)} \) are also positive.

2. Positivity of \( A_i^{(n)} \). We consider the Legendre function of second kind

\[ Q_n(x) = \frac{1}{2} \int_{-1}^{1} \frac{P_n(t)}{x - t} dt, \quad n \geq 1, \]

defined for any \( x \) in the complex plane cut along the segment \([-1, 1]\); we introduce the function

\[ \bar{Q}_n(x) = \frac{1}{2} \lim_{\epsilon \to +0} [Q_n(x + i\epsilon) + Q_n(x - i\epsilon)], \]

which is analytic on \((-1, 1)\). It is known [5, p. 78] that

Received April 21, 1977.


*Work performed under the auspices of the Italian Research Council.

Copyright © 1978, American Mathematical Society

243
From (2), (3) and (4), and recalling Lebesgue's convergence theorem, it then follows that at the zeros \( \xi_i^{(n)} \), \( i = 1, \ldots, n \), of \( P_n(x) \) we have

\[
Q_n(\xi_i^{(n)}) = \frac{1}{2} \int_{-1}^{1} \frac{P_n(t)}{\xi_i^{(n)} - t} \, dt.
\]

Let now

\[
E_{n+1}(\cos \phi) = \lambda_0 \cos(n + 1)\phi + \lambda_1 \cos(n - 1)\phi + \cdots + \begin{cases} 
\lambda_{n/2} \cos \phi, & n \text{ even}, \\
\frac{1}{2} \lambda_{(n+1)/2}, & n \text{ odd}, 
\end{cases}
\]

and

\[
e_{n+1}(\phi) = \lambda_0 \sin(n + 1)\phi + \lambda_1 \sin(n - 1)\phi + \cdots + \begin{cases} 
\lambda_{n/2} \sin \phi, & n \text{ even}, \\
0, & n \text{ odd}, 
\end{cases}
\]

where \( x = \cos \phi, 0 < \phi < \pi \), and, as known \[4\], \( \lambda_0 = (2n + 1)!/(2^n n!)^2 \). Then, Szegö in his paper \[4, p. 507\] gives the following inequality

\[
\left| E_{n+1}(\xi_i^{(n)}) \right| > \left| \overline{Q}_n(\xi_i^{(n)}) \right|^{-1}, \quad i = 1, \ldots, n.
\]

We are now ready to prove the following

**Theorem.** The weights \( A_i^{(n)} \) and \( B_j^{(n)} \) of the extended Gauss-Legendre rules are always positive.

**Proof.** The positivity of \( B_j^{(n)} \) has already been proved in \[2\]. In that paper, the following expression for the weights \( A_i^{(n)} \) has also been given

\[
A_i^{(n)} = H_i^{(n)} - \frac{h_n}{k_n \left| P_n'(\xi_i^{(n)}) \right| \left| q_{n+1}(\xi_i^{(n)}) \right|}, \quad i = 1, \ldots, n,
\]

where \( H_i^{(n)} = 2 \left| \overline{Q}_n(\xi_i^{(n)}) \right| / \left| P_n'(\xi_i^{(n)}) \right| \) are the weights of the \( n \)-point Gauss-Legendre rule, \( h_n = 2/(2n + 1) \), \( k_n = (2n)!/(2^n n!)^2 \) and \( q_{n+1}(x) = 1/(2^n \lambda_0) E_{n+1}(x) \). Recalling (7), from (8) we have

\[
A_i^{(n)} > H_i^{(n)} \left( 1 - \frac{h_n}{k_n} 2^{n-1} \lambda_0 \right) = 0,
\]

which proves the theorem.

What follows is an immediate consequence (see for example \[5, Theorem 15.2.2\]) of the theorem we have just proved.

**Corollary.** The quadrature process defined by (1) is convergent for every function \( f(x) \) which is Riemann-integrable in \([-1, 1]\), i.e., \( \lim_{n \to \infty} R_n(f) = 0 \).

**Remark.** In his paper, Szegö derives, although not explicitly stated, the analogue of (6) for rules of type (1) with a weight function of the form \( (1 - x^2)^{\nu - \frac{3}{2}} \), when
0 < μ < 1. In a way very similar to the Legendre case, it may then be shown that the weights of that type of rules are positive, too. For μ = 0, 1 see [2].

Istituto di Calcoli Numerici
Università di Torino
10123 Torino, Italy