Odd Integers $N$ With Five Distinct Prime Factors
for Which $2 - 10^{-12} < \sigma(N)/N < 2 + 10^{-12}$

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Abstract. We make a table of odd integers $N$ with five distinct prime factors for which $2 - 10^{-12} < \sigma(N)/N < 2 + 10^{-12}$, and show that for such $N$ $|\sigma(N)/N - 2| > 10^{-14}$. Using this inequality, we prove that there are no odd perfect numbers, no quasiperfect numbers and no odd almost perfect numbers with five distinct prime factors. We also make a table of odd primitive abundant numbers $N$ with five distinct prime factors for which $2 < \sigma(N)/N < 2 + 2/10^{10}$.

1. A positive integer $N$ is called perfect, quasiperfect (QP), or almost perfect according as $\sigma(N) = 2N$, $2N + 1$, or $2N - 1$, respectively, where $\sigma(N)$ is the sum of the positive divisors of $N$. While twenty-four even perfect numbers are known, no odd perfect (OP) numbers, no QP numbers, and no almost perfect numbers except a power of 2 are known.

In this paper we make a table of odd integers $N$ with five distinct prime factors for which

$$2 - 10^{-12} < \sigma(N)/N < 2 + 10^{-12},$$

and we show that for such $N$

$$|\sigma(N)/N - 2| > 10^{-14}.$$ 

Using this inequality, we prove that there are no OP, QP, or odd almost perfect (OAP) numbers with five distinct prime factors.

$N$ is called primitive abundant if $N$ is abundant ($\sigma(N) > 2N$) and every proper divisor $M$ of $N$ is deficient ($\sigma(M) < 2M$). In 1913 Dickson [4] published a table of odd primitive abundant numbers with less than five distinct prime factors. In this paper we also make a table of odd primitive abundant numbers $N$ with five distinct prime factors for which

$$2 < \sigma(N)/N < 2 + 2/10^{10}.$$

2. Throughout this paper we let $N = \prod_{i=1}^{r} p_i^{q_i}$ where $3 < p_1 < \cdots < p_r$ are primes and $q_i$'s are positive integers. $p_i^{q_i}$ is called a component of $N$.

We define

$$a(p) = \min\{a|p^{a+1} > 10^{12}\},$$

$$\omega(N) = r,$$

$$S(N) = \sigma(N)/N = \prod_{i=1}^{r} (p_i^{q_i+1} - 1)/p_i^{q_i}(p_i - 1),$$

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\[ A(N) = \left[ \prod_{a_{i} < a(p_i)} S(p_i^{a_i}) \right] \left[ \prod_{a_{i} \geq a(p_i)} S(p_i^{a(p_i)}) \right], \]

\[ B(N) = \left[ \prod_{a_{i} < a(p_i)} S(p_i^{a_i}) \right] \left[ \prod_{a_{i} \geq a(p_i)} p_i/(p_i - 1) \right], \]

\[ L(p^a) = \begin{cases} 
[10^{12} \log S(p^a)]/10^{12} & \text{if } a < a(p), \\
[10^{12} \log p/(p - 1)]/10^{12} & \text{if } a \geq a(p), 
\end{cases} \]

where \([ \ ]\) is the greatest integer function. We note that if \( p, q \) are primes with \( p > q \) and \( a, b \) are positive integers then

\[ S(p^a) = (p^{a+1} - 1)/p^a(p - 1) < p/(p - 1) = \lim_{a \to \infty} S(p^a) \leq (q + 1)/q \leq S(q^b), \]

and so \( L(p^a) \leq L(q^b) \) and \( A(N) \leq S(N) \leq B(N) \). Hence, we have

**Lemma 1.** (a) If \( A(N) > 2 - 10^{-12} \) and \( B(N) < 2 + 10^{-12} \), \( N \) satisfies (1).

(b) If \( A(N) < 2 - 10^{-12} \leq B(N) < 2 + 10^{-12}, \) some \( N \) satisfies (1).

(c) If \( 2 - 10^{-12} < A(N) < 2 + 10^{-12} \leq B(N) \), some \( N \) satisfies (1).

(d) If \( A(N) < 2 - 10^{-12} \) and \( 2 + 10^{-12} < B(N) \), some \( N \) may satisfy (1).

(e) If \( 2 + 10^{-12} < A(N) \) or \( B(N) < 2 - 10^{-12} \), \( N \) does not satisfy (1).

In Lemmas 2 through 5 we assume that \( N \) satisfies (1) and \( \omega(N) = 5 \).

**Lemma 2.**

\[ 0.6931471805544 < \sum_{i=1}^{5} L(p_i^{b_i}) < 0.6931471805655, \]

where \( b_i = \min\{a_i, a(p_i)\} \).

**Proof.** Suppose \( p^a \) is a component of \( N \). If \( a < a(p) \), then

\[ |\log S(p^a) - L(p^a)| < 10^{-12}. \]

If \( a \geq a(p) \), then \( p^{a+1} > 10^{12} \) and

\[ 10^{-12} > \log p/(p - 1) - L(p^a) > \log S(p^a) - L(p^a) \geq \log S(p^a) - \log p/(p - 1) = \log (1 - 1/p^{a+1}) = - \sum_{i=1}^{\infty} 1/i(p^{a+1})^i > -1/(p^{a+1} - 1) \geq -10^{-12}. \]

Hence

\[ |\log S(p^a) - L(p^a)| < 10^{-12}. \]

Since (1) holds,

\[ 0.6931471805544 < \log(2 - 10^{-12}) - 5/10^{12} < \sum_{i=1}^{5} \log S(p_i^{a_i}) - 5/10^{12} < \sum_{i=1}^{5} L(p_i^{b_i}) < \sum_{i=1}^{5} \log S(p_i^{a_i}) + 5/10^{12} < \log(2 + 10^{-12}) + 5/10^{12} < 0.6931471805655. \]
Lemma 3. \( p_1 = 3, p_2 \leq 11 \) and \( p_3 \leq 41 \).

Proof. Lemma 3 follows from the following inequalities:

\[
\begin{align*}
\frac{5}{4} &\quad \frac{7}{6} &\quad \frac{11}{10} &\quad \frac{13}{12} &\quad \frac{17}{16} < 2 - 10^{-12}, \\
\frac{3}{2} &\quad \frac{13}{12} &\quad \frac{17}{16} &\quad \frac{19}{15} &\quad \frac{23}{22} < 2 - 10^{-12}, \\
\frac{3}{2} &\quad \frac{5}{4} &\quad \frac{43}{42} &\quad \frac{47}{46} &\quad \frac{53}{52} < 2 - 10^{-12}. \quad \text{Q.E.D.}
\end{align*}
\]

Lemma 4. \( p_4 < 5000 \).

Proof. Suppose \( N \) satisfies (1) and \( p_4 \geq 5003 \). Then

\[
\begin{align*}
0 &\leq L(p_5^{b_5}) \leq L(p_4^{b_4}) < \log S(p_4^{b_4}) + 10^{-12} \\
&< \log p_4/(p_4 - 1) + 10^{-12} < 1/(p_4 - 1) + 10^{-12} \\
&< 0.0002.
\end{align*}
\]

Hence by (3)

\[
(4) \quad 0.69274 < \sum_{i=1}^{3} L(p_i^{b_i}) < 0.69315.
\]

A computer (PDP11 at the University of Toledo) was used to find \( \Pi_{i=1}^{3} p_i^{b_i} \) satisfying (4), but there were none. Q.E.D.

Similarly, we can prove

Lemma 5. \( p_5 < 3000000 \), or \( \Pi_{i=1}^{5} p_i^{b_i} = 3756172233 \) and \( 36549767 \leq p_5 \leq 36551083 \).

The computer was used to find \( N = \Pi_{i=1}^{5} p_i^{b_i} \) satisfying \( a_i = a(p_i) \), Lemmas 3, 4, 5, and Lemma 2 or Lemma 1(b), (c), (d), with the result given in Table 1.

Lemma 6. Suppose \( N = \Pi_{i=1}^{5} p_i^{b_i} \) and \( M = \Pi_{i=1}^{5} p_i^{b_i} \) where \( b_i = \min \{ a_i, a(p_i) \} \).

If \( M = 3^{2}5^{12}7^{6}257^{4}65521, \) \( |S(N) - 2| > 1/10^{-13} \); \( |S(N) - 2| > 2/10^{-13} \); \( |S(N) - 2| > 3/10^{-13} \);

if \( M = 3^{8}5^{14}7^{3}251\cdot1884529, \) \( |S(N) - 2| > 2/10^{-14} \); \( |S(N) - 2| > 3/10^{-13} \);

if \( M = 3^{8}5^{8}17^{9}269^{4}4153^{3}, \) \( |S(N) - 2| > 4/10^{-14} \); \( |S(N) - 2| > 3/10^{-13} \);

if \( M = 3^{8}5^{8}17^{9}269^{4}4153^{3}, \) \( |S(N) - 2| > 4/10^{-14} \); \( |S(N) - 2| > 3/10^{-13} \);

In all other cases \( |S(N) - 2| > 10^{-13} \).

Proof. The first part of Lemma 6 follows from the following inequalities:

\[
\begin{align*}
S(3^{2}3^{5}127^{6}257^{4}65521) &< 2 - 5/10^{13}, \\
S(3^{2}3^{5}127^{6}257^{4}65521) &> 2 + 1/10^{12}, \\
S(3^{8}5^{14}7^{3}251\cdot1884529) &< 2 - 2/10^{14}, \\
S(3^{8}5^{9}7^{3}251\cdot1579769) &< 2 - 4/10^{13}, \\
S(3^{8}5^{9}7^{3}251\cdot1579769^{2}) &> 2 + 3/10^{13}, \\
S(3^{8}5^{8}269^{4}) &< 2 + 4/10^{14}, \\
S(3^{8}5^{8}17^{9}269^{4}4153^{3}) &> 2 + 3/10^{13}, \\
S(3^{8}5^{8}17^{2}233\cdot36550379) &> 2 + 5/10^{14},
\end{align*}
\]
and
\[ S(3^75^617^2233) \frac{36550429}{36550428} < 2 - 10^{-14}. \]

Suppose \(|S(N) - 2| < 10^{-13}\). Then (1) holds, and so \(N\) is given in Table 1; however, for every \(N\) in Table 1 except for those given above \(S(N) \leq B(N) < 2 - 10^{-13}\), or \(S(N) > A(N) > 2 + 10^{-13}\). Q.E.D.

We have proved

**Theorem.** If \(N\) is an odd integer with \(\omega(N) = 5\), \(|\sigma(N)/N - 2| > 10^{-14}\).

3. We used a similar method to find odd primitive abundant numbers \(N = \prod_{i=1}^{5} p_i^{a_i}\) for which (2) holds, with the result given in Table 2 in the microfiche. Table 2 includes odd primitive abundant numbers \(N\) with \(\omega(N) = 5\) one of whose component \(p^a\) is greater than \(10^{10}\); for, letting \(M = N/p^a\), we have

\[
2 < \frac{\sigma(N)/N = \sigma(M)\sigma(p^a)/Mp^a = \sigma(M)(p\sigma(p^{a-1}) + 1)/Mp^a}{\sigma(M)p^{a-1}/Mp^{a-1} + \sigma(M)/Mp^a < 2 + 2/10^{10}},
\]
showing that (2) holds.

4. Suppose \(N\) is an odd integer such that \(\sigma(N) = 2N + A\). If \(|A/N| < 10^{-14}\), then by our Theorem \(\omega(N) \geq 6\). We give three examples of such \(N\).

Suppose \(N\) is OP. Sylvester (1888), Dickson (1913), and Kanold (1949) proved that \(\omega(N) \geq 5\). From our Theorem we have

**Proposition 1.** If \(N\) is OP, \(\omega(N) \geq 6\).

This fact was also proved by Gradšteiın (1925), Kühnel (1949) and Webber (1951). Pomerance [1] (1972) and Robbins (1972) proved that \(\omega(N) \geq 7\), and Hagis [2] proved that \(\omega(N) \geq 8\).

**Proposition 2.** If \(N\) is QP, \(\omega(N) \geq 6\).

**Proof.** By [3] if \(N\) is QP, then \(N\) is an odd perfect square, \(\omega(N) \geq 5\) and \(N > 10^{20}\). Hence \(2 < S(N) = 2 + 2/N < 2 + 10^{-20}\), and so by Theorem \(\omega(N) \geq 6\). Q.E.D.

**Lemma 7.** If \(N\) is OAP, \(pN\) is primitive abundant for some \(p \mid N\).

**Proof.** Suppose \(N = \prod_{i=1}^{a} p_i^{a_i}\) is OAP, and choose \(j\) so that \(\sigma(p_i^{a_j}) > \sigma(p_i^{a_i})\) for every \(i\). Letting \(p = p_j\), \(a = a_j\) and \(L = N/p^a\), we have

\[
2p^aL - 1 = \sigma(N) = \sigma(p^a)\sigma(L) = \sigma(L) + p\sigma(p^{a-1})\sigma(L).
\]

Hence \(p \mid \sigma(L) + 1\). If \(p = \sigma(L) + 1\), then

\[
\sum_{i=1}^{a+1} p^i = \sigma(p^a)p = \sigma(p^a)\sigma(L) + \sigma(p^a)
\]

\[
= \sigma(N) + \sigma(p^a) = 2p^aL - 1 + \sigma(p^a) = 2p^aL + \sum_{i=1}^{a} p^i,
\]
or \(p^{a+1} = 2p^aL\), showing that \(N = 2^a\). Since \(N\) is OAP, \(p \neq \sigma(L) + 1\), and so \(p < \sigma(L)\) because \(p \mid \sigma(L) + 1\). Then

\[
\sigma(pN) = \sigma(p^{a+1})\sigma(L) = (1 + p\sigma(p^a))\sigma(L)
\]

\[
= \sigma(L) + p\sigma(N) = \sigma(L) + 2pN - p > 2pN,
\]
showing that \(pN\) is abundant.
Suppose $M$ is a proper divisor of $pN$. If $p^{a+1} \nmid M$, then $M$ is a divisor of $N$, and $M$ is deficient because

$$S(M) \leq S(N) = 2 - 1/N < 2.$$ 

Suppose $p^{a+1} \mid M$. Then for some $k$, $p^{ak} \nmid M$. Letting $q = p_k$ and $b = a_k$, we have $\sigma(p^a) \geq \sigma(q^b)$, or

$$\sum_{i=1}^{b} q^i \leq \sum_{i=1}^{a} p^i < \sum_{i=1}^{a+1} p^i.$$ 

Hence

$$(1/p^{a+1}) \sum_{i=0}^{b-1} q^{-i} < (1/q^b) \sum_{i=0}^{a} p^{-i},$$

and by adding $\sum_{i=0}^{a} p^{-i} \sum_{i=0}^{b-1} q^{-i}$ to both sides we obtain

$$\sum_{i=1}^{a+1} p^{-i} \sum_{i=0}^{b-1} q^{-i} < \sum_{i=0}^{a} p^{-i} \sum_{i=0}^{b} q^{-i},$$

or $S(p^{a+1})S(q^{b-1}) < S(p^a)S(q^b)$. Then

$$S(M) < S(p^{a+1})S(q^{b-1}) \prod_{i \neq j,k} S(p_i^{a_i})$$

$$< S(p^a)S(q^b) \prod_{i \neq j,k} S(p_i^{a_i}) = S(N) < 2,$$

showing that $M$ is deficient. Q.E.D.

**Lemma 8.** If $N = \prod_{i=1}^{r} p_i^{a_i}$ is OAP, $a_i$ is even. If $p_1 = 3$, $a_1 \geq 12$.

**Proof.** Suppose $N$ is OAP, $p_i$ is a component of $N$, $q$ is a prime and $q \mid \sigma(p^a)$. Since $\sigma(N) = 2N - 1$ is odd and $\sigma(p^a) \mid \sigma(N)$, $\sigma(p^a) = \sum_{i=0}^{a} p^i$ is odd. Hence $a$ is even. Since $q \mid 2\sigma(N) = 4N - 2$ and $4N$ is a perfect square, $(2 \mid q) = 1$, where $(2 \mid q)$ is the Legendre symbol, and so $q \equiv 1$ or 7 (mod 8) because $(2 \mid q) = (-1)^{(q^2-1)/8}$. Also $\sigma(p^a) \equiv 1$ or 7 (mod 8), for, otherwise, $\sigma(p^a)$ would have a prime factor $\equiv 3$ or 5 (mod 8).

Suppose $p = 3$ and $a = 2e$. Then $\sigma(3^{2e}) = 1 + 4e \equiv 1$ or 7 (mod 8), or $e \equiv 0$ (mod 2). Hence $a = 4, 8, 12, \ldots$; however, $a \neq 4$ or 8 because $11 \mid \sigma(3^8), 11 \equiv 3$ (mod 8), $13 \mid \sigma(3^8)$ and $13 \equiv 5$ (mod 8). Q.E.D.

**Proposition 4.** If $N$ is OAP, $\omega(N) \geq 6$.

**Proof.** Suppose $N = \prod_{i=1}^{r} p_i^{a_i}$ is OAP. Then by Lemma 7 $pN$ is primitive abundant for some $p \mid N$. If $3 \nmid N$, $\omega(N) \geq 7$, for, otherwise,

$$2 < S(pN) < \prod_{i=1}^{r} \frac{p_i}{p_i - 1} \leq \frac{5}{4} \frac{7}{6} \frac{11}{10} \frac{13}{12} \frac{17}{16} \frac{19}{18} < 2.$$ 

Suppose $3 \mid N$. Then $3^{12} \mid pN$ by Lemma 8. According to the table of odd primitive abundant numbers $M$ with fewer than five distinct prime factors in [4] $3^{12} \nmid M$. 

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Hence $\omega(N) \geq 5$, and $N \geq 3^{12}5^27^211^213^2 > 10^{13}$. Then $2 > S(N) = 2 - 1/N > 2 - 10^{-13}$, and by Lemma 6 $\omega(N) \geq 6$. Q.E.D.

For other results on QP and OAP see [3], [5], [6], [7] and [8]. Computer time for Tables 1 and 2 was over four hours.

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**Table 1**

$N = \prod_{i=1}^{5} p_i^{a_i}$ for which $2 - 10^{-12} < o(N)/N < 2 + 10^{-12}\, (a)$

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**Note:**
(a) If $b_i = a(p_i)$ and $c > 0$, $Np_i^c$ also satisfies (1). See Lemma 1(a).
(b) See Lemma 1(b).
(c) See Lemma 1(c).
(d) See Lemma 1(d).
(e) $36549767 < p_5 \leq 36551083$. 

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5. L. E. DICKSON, "Finiteness of the odd perfect and primitive abundant numbers with \( n \) distinct prime factors," *Amer. J. Math.*, v. 35, 1913, pp. 413–422.