Odd Integers $N$ With Five Distinct Prime Factors for Which $2 - 10^{-12} < \sigma(N)/N < 2 + 10^{-12}$

By Masao Kishore*

Abstract. We make a table of odd integers $N$ with five distinct prime factors for which $2 - 10^{-12} < \sigma(N)/N < 2 + 10^{-12}$, and show that for such $N$ $|\sigma(N)/N - 2| > 10^{-14}$. Using this inequality, we prove that there are no perfect numbers, no quasiperfect numbers and no odd almost perfect numbers with five distinct prime factors. We also make a table of odd primitive abundant numbers $N$ with five distinct prime factors for which $2 < \sigma(N)/N < 2 + 2/10^{10}$.

1. A positive integer $N$ is called perfect, quasiperfect (QP), or almost perfect according as $\sigma(N) = 2N$, $2N + 1$, or $2N - 1$, respectively, where $\sigma(N)$ is the sum of the positive divisors of $N$. While twenty-four even perfect numbers are known, no odd perfect (OP) numbers, no QP numbers, and no almost perfect numbers except a power of 2 are known.

In this paper we make a table of odd integers $N$ with five distinct prime factors for which

\[ 2 - 10^{-12} < \sigma(N)/N < 2 + 10^{-12}, \]

and we show that for such $N$

\[ |\sigma(N)/N - 2| > 10^{-14}. \]

Using this inequality, we prove that there are no OP, QP, or odd almost perfect (OAP) numbers with five distinct prime factors.

$N$ is called primitive abundant if $N$ is abundant ($\sigma(N) > 2N$) and every proper divisor $M$ of $N$ is deficient ($\sigma(M) < 2M$). In 1913 Dickson [4] published a table of odd primitive abundant numbers with less than five distinct prime factors. In this paper we also make a table of odd primitive abundant numbers $N$ with five distinct prime factors for which

\[ 2 < \sigma(N)/N < 2 + 2/10^{10}. \]

2. Throughout this paper we let $N = \Pi_{i=1}^{r} p_i^{a_i}$ where $3 < p_1 < \cdots < p_r$ are primes and $a_i$'s are positive integers. $p_i^{a_i}$ is called a component of $N$.

We define

\[ \sigma(p) = \min\{a|p^{a+1} > 10^{12}\}, \]

\[ \omega(N) = r, \]

\[ S(N) = \sigma(N)/N = \prod_{i=1}^{r} (p_i^{a_i+1} - 1)/p_i^{a_i}(p_i - 1), \]

Received December 6, 1976; revised March 23, 1977.


*This paper is a part of the author's doctoral dissertation which was submitted to Princeton University in August 1977 and directed by Professor J. Chidambaramsamy of the University of Toledo.

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\[ A(N) = \left[ \prod_{a_1 < a(p_i)} S(p_i^{a_i}) \right] \left[ \prod_{a_1 \geq a(p_i)} S(p_i^{a_i(p_i)}) \right], \]

\[ B(N) = \left[ \prod_{a_1 < a(p_i)} S(p_i^{a_i}) \right] \left[ \prod_{a_1 \geq a(p_i)} p_i / (p_i - 1) \right], \]

\[ L(p^a) = \begin{cases} \left\lfloor \frac{10^{12} \log S(p^a)}{10^{12}} \right\rfloor & \text{if } a < a(p), \\
\left\lfloor \frac{10^{12} \log p_i / (p_i - 1)}{10^{12}} \right\rfloor & \text{if } a \geq a(p), \end{cases} \]

where \( \lfloor \rfloor \) is the greatest integer function. We note that if \( p, q \) are primes with \( p > q \) and \( a, b \) are positive integers then

\[ S(p^a) = \frac{(p^a + 1 - 1)}{p^a(p - 1)} < p/(p - 1) = \lim_{a \to \infty} S(p^a) \leq (q + 1)/q \leq S(q^b), \]

and so \( L(p^a) \leq L(q^b) \) and \( A(N) \leq S(N) \leq B(N) \). Hence, we have

**Lemma 1.** (a) If \( A(N) > 2 - 10^{-12} \) and \( B(N) < 2 + 10^{-12} \), \( N \) satisfies (1). 
(b) If \( A(N) < 2 - 10^{-12} < B(N) < 2 + 10^{-12} \), some \( N \) satisfies (1). 
(c) If \( 2 - 10^{-12} < A(N) < 2 + 10^{-12} \leq B(N) \), some \( N \) satisfies (1). 
(d) If \( A(N) < 2 - 10^{-12} \) and \( 2 + 10^{-12} < B(N) \), some \( N \) may satisfy (1). 
(e) If \( 2 + 10^{-12} < A(N) \) or \( B(N) < 2 - 10^{-12} \), \( N \) does not satisfy (1). 

In Lemmas 2 through 5 we assume that \( N \) satisfies (1) and \( \omega(N) = 5 \). 

**Lemma 2.**

\[ 0.6931471805544 < \sum_{i=1}^{5} L(p_i^{b_i}) < 0.6931471805655, \]

where \( b_i = \min \{ a_i, a(p_i) \} \).

**Proof.** Suppose \( p^a \) is a component of \( N \). If \( a < a(p) \), then

\[ |\log S(p^a) - L(p^a)| < 10^{-12}. \]

If \( a \geq a(p) \), then \( p^{a + 1} > 10^{12} \) and

\[ 10^{-12} > \log p_i / (p_i - 1) - L(p^a) > \log S(p^a) - L(p^a) \geq \log S(p^a) - \log p_i / (p_i - 1) \]

\[ = \log \left( 1 - 1/p^{a + 1} \right) = - \sum_{i=1}^{\infty} 1/i(p^{a + 1}) > -1/(p^{a + 1} - 1) \geq -10^{-12}. \]

Hence

\[ |\log S(p^a) - L(p^a)| < 10^{-12}. \]

Since (1) holds,

\[ 0.6931471805544 < \log(2 - 10^{-12}) - 5/10^{12} \]

\[ < \sum_{i=1}^{5} \log S(p_i^{a_i}) - 5/10^{12} < \sum_{i=1}^{5} L(p_i^{b_i}) \]

\[ < \sum_{i=1}^{5} \log S(p_i^{a_i}) + 5/10^{12} < \log(2 + 10^{-12}) + 5/10^{12} \]

\[ < 0.6931471805655. \quad Q.E.D. \]
Lemma 3. $p_1 = 3$, $p_2 \leq 11$ and $p_3 \leq 41$.

Proof. Lemma 3 follows from the following inequalities:

\[
\begin{align*}
\frac{5}{4} \frac{7}{6} \frac{11}{10} \frac{13}{12} \frac{17}{16} &< 2 - 10^{-12}, \\
\frac{3}{2} \frac{13}{12} \frac{17}{16} \frac{19}{18} \frac{23}{22} &< 2 - 10^{-12}, \\
\frac{3}{2} \frac{5}{4} \frac{43}{42} \frac{47}{46} \frac{53}{52} &< 2 - 10^{-12}. \quad \text{Q.E.D.}
\end{align*}
\]

Lemma 4. $p_4 < 5000$.

Proof. Suppose $N$ satisfies (1) and $p_4 \geq 5003$. Then

\[
\begin{align*}
0 &< L(p_5^{b_5}) < L(p_4^{b_4}) < \log S(p_4^{b_4}) + 10^{-12} \\
&< \log p_4/(p_4 - 1) + 10^{-12} < 1/(p_4 - 1) + 10^{-12} \\
&< 0.0002.
\end{align*}
\]

Hence by (3)

\[0.69274 < \sum_{i=1}^{3} L(p_i^{b_i}) < 0.69315.\]

A computer (PDP11 at the University of Toledo) was used to find $\prod_{i=1}^{4} p_i^{b_i}$ satisfying (4), but there were none. Q.E.D.

Similarly, we can prove

Lemma 5. $p_5 < 3000000$, or $\prod_{i=1}^{5} p_i^{b_i} = 3^5 5^3 17^2 233$ and $36549767 \leq p_5 \leq 36551083$.

The computer was used to find $N = \prod_{i=1}^{5} p_i^{b_i}$ satisfying $a_i \leq a(p_i)$, Lemmas 3, 4, 5, and Lemma 2 or Lemma 1(b), (c), (d), with the result given in Table 1.

Lemma 6. Suppose $N = \prod_{i=1}^{5} p_i^{b_i}$ and $M = \prod_{i=1}^{5} p_i^{b_i}$ where $b_i = \min \{a_i, a(p_i)\}$.

If $M = 3^{23} 5^{12} 17^6 257^4 65521$, $|S(N) - 2| > 5/10^{13}$;

if $M = 3^{514} 17^3 251 \cdot 1884529$, $|S(N) - 2| > 2/10^{14}$;

if $M = 3^8 5^9 17^3 251 \cdot 1579769$, $|S(N) - 2| > 3/10^{13}$;

if $M = 3^8 5^8 17^9 269^4 4153^3$, $|S(N) - 2| > 4/10^{14}$;

if $\prod_{i=1}^{4} p_i^{b_i} = 3^7 5^5 17^2 233$, $|S(N) - 2| > 10^{-14}$.

In all other cases $|S(N) - 2| > 10^{-13}$.

Proof. The first part of Lemma 6 follows from the following inequalities:

\[
\begin{align*}
S(3^{23} 5^{12} 17^6 257^4 65521) &< 2 - 5/10^{13}, \\
S(3^{23} 5^{12} 17^6 257^4 65521) &> 2 + 1/10^{12}, \\
S(3^8 5^{14} 17^3 251) 1884529/1884528 &< 2 - 2/10^{14}, \\
S(3^8 5^9 17^3 251 \cdot 1579769) &< 2 - 4/10^{13}, \\
S(3^8 5^9 17^3 251 \cdot 1579769^2) &> 2 + 3/10^{13}, \\
S(3^8 5^8 269^4 ) 17/16 \cdot 4153/4152 &< 2 - 4/10^{14}, \\
S(3^8 5^8 17^9 269^4 4153^3) &> 2 + 3/10^{13}, \\
S(3^7 5^5 17^2 233 \cdot 36550379) &> 2 + 5/10^{14},
\end{align*}
\]
Suppose \( |S(N) - 21| < 10^{-13} \). Then (1) holds, and so \( N \) is given in Table 1; however, for every \( N \) in Table 1 except for those given above \( S(N) = B(N) < 2 - 10^{-13} \), or \( S(N) = A(N) > 2 + 10^{-13} \). Q.E.D.

We have proved

**THEOREM.** If \( N \) is an odd integer with \( \omega(N) = 5 \), \( |\sigma(N)/N - 21| > 10^{-14} \).

3. We used a similar method to find odd primitive abundant numbers \( N = \prod_{i=1}^{5} p_i^{a_i} \) for which (2) holds, with the result given in Table 2 in the microfiche.

Table 2 includes odd primitive abundant numbers \( N \) with \( \omega(N) = 5 \) one of whose component \( p^a \) is greater than \( 10^{10} \); for, letting \( M = N/p^a \), we have

\[
2 < \sigma(N)/N = \sigma(M)\sigma(p^a)/M\sigma(p^a) = \sigma(M)(\sigma(p^{a-1}) + 1)/M\sigma(p^a) = \sigma(M\sigma(p^{a-1}))/M\sigma(p^{a-1}) + \sigma(M)/M\sigma(p^a) < 2 + 2/10^{10},
\]

showing that (2) holds.

4. Suppose \( N \) is an odd integer such that \( \sigma(N) = 2N + A \). If \( |A/N| < 10^{-14} \), then by our Theorem \( \omega(N) \geq 6 \). We give three examples of such \( N \).

Suppose \( N \) is OP. Sylvester (1888), Dickson (1913), and Kanold (1949) proved that \( \omega(N) \geq 5 \). From our Theorem we have

**PROPOSITION 1.** If \( N \) is OP, \( \omega(N) \geq 6 \).

This fact was also proved by Gradšteīn (1925), Kühnel (1949) and Webber (1951). Pomerance [1] (1972) and Robbins (1972) proved that \( \omega(N) \geq 7 \), and Hagis [2] proved that \( \omega(N) \geq 8 \).

**PROPOSITION 2.** If \( N \) is QP, \( \omega(N) \geq 6 \).

**Proof.** By [3] if \( N \) is QP, then \( N \) is an odd perfect square, \( \omega(N) \geq 5 \) and \( N > 10^{20} \). Hence \( 2 < S(N) = 2 + 1/N < 2 + 10^{-20} \), and so by Theorem \( \omega(N) \geq 6 \). Q.E.D.

**LEMMA 7.** If \( N \) is OAP, \( pN \) is primitive abundant for some \( p \nmid N \).

**Proof.** Suppose \( N = \prod_{i=1}^{a} p_i^{a_i} \) is OAP, and choose \( j \) so that \( \sigma(p_i^{a_i}) > \sigma(p_i^{a_i}) \) for every \( i \). Letting \( p = p_j \), \( a = a_j \) and \( L = N/p^a \), we have

\[
2p^aL - 1 = \sigma(N) = \sigma(p^a)\sigma(L) = (1 + p\sigma(p^{a-1}))\sigma(L) = \sigma(L) + p\sigma(p^{a-1})\sigma(L).
\]

Hence \( p \mid \sigma(L) + 1 \). If \( p = \sigma(L) + 1 \), then

\[
\sum_{i=1}^{a+1} p_i = \sigma(p^a)p = \sigma(p^a)\sigma(L) + \sigma(p^a)
\]

\[
= \sigma(N) + \sigma(p^a) = 2p^aL - 1 + \sigma(p^a) = 2p^aL + \sum_{i=1}^{a} p_i,
\]

or \( p^{a+1} = 2p^aL \), showing that \( N = 2^a \). Since \( N \) is OAP, \( p \neq \sigma(L) + 1 \), and so \( p < \sigma(L) \) because \( p \mid \sigma(L) + 1 \). Then

\[
\sigma(pN) = \sigma(p^{a+1})\sigma(L) = (1 + p\sigma(p^{a}))\sigma(L) = \sigma(L) + p\sigma(N) = \sigma(L) + 2pN - p > 2pN,
\]

showing that \( pN \) is abundant.
Suppose \( M \) is a proper divisor of \( p \mathbb{N} \). If \( p^{a+1} \nmid M \), then \( M \) is a divisor of \( \mathbb{N} \), and \( M \) is deficient because

\[
S(M) = S(\mathbb{N}) = 2 - 1/\mathbb{N} < 2.
\]

Suppose \( p^{a+1} \mid M \). Then for some \( k, p^a \nmid M \). Letting \( q = p_k \) and \( b = a_k \), we have

\[
\sigma(p^a) \geq \sigma(q^b),
\]

or

\[
\sum_{i=1}^{b} q^i \leq \sum_{i=1}^{a} p^i < \sum_{i=1}^{a+1} p^i.
\]

Hence

\[
(1/p^{a+1}) \sum_{i=0}^{b-1} q^{-i} < (1/q^b) \sum_{i=0}^{a} p^{-i},
\]

and by adding \( \sum_{i=0}^{b-1} q^{-i} \) to both sides we obtain

\[
\sum_{i=0}^{a+1} p^{-i} \sum_{i=0}^{b-1} q^{-i} < \sum_{i=0}^{a} p^{-i} \sum_{i=0}^{b} q^{-i},
\]

or \( S(p^{a+1}) S(q^{b-1}) < S(p^a) S(q^b) \). Then

\[
S(M) < S(p^{a+1}) S(q^{b-1}) \prod_{i \neq j, k} S(p_i^i) < S(p^a) S(q^b) \prod_{i \neq j, k} S(p_i^i) = S(\mathbb{N}) < 2,
\]

showing that \( M \) is deficient. Q.E.D.

**Lemma 8.** If \( N = \Pi_{i=1}^r p_i^{a_i} \) is OAP, \( a_i \) is even. If \( p_1 = 3, a_1 \geq 12 \).

**Proof.** Suppose \( N \) is OAP, \( p_1 \) is a component of \( N \), \( q \) is a prime and \( q \mid \sigma(p^a) \).

Since \( \sigma(N) = 2N - 1 \) is odd and \( \sigma(p^a) \mid \sigma(N), \sigma(p^a) = \sum_{i=0}^{a} p_i^i \) is odd. Hence \( a \) is even.

Since \( q \mid 2\sigma(N) = 4N - 2 \) and \( 4N \) is a perfect square, \( (2 \mid q) = 1 \), where \( (2 \mid q) \) is the Legendre symbol, and so \( q \equiv 1 \) or \( 7 \) (mod 8) because \( (2 \mid q) = (-1)^{(q^2-1)/8} \). Also \( \sigma(p^a) \equiv 1 \) or \( 7 \) (mod 8), for, otherwise, \( \sigma(p^a) \) would have a prime factor \( \equiv 3 \) or 5 (mod 8).

Suppose \( p = 3 \) and \( a = 2e \). Then \( \sigma(3^{2e}) \equiv 1 + 4e \equiv 1 \) or \( 7 \) (mod 8), or \( e \equiv 0 \) (mod 2). Hence \( a = 4, 8, 12, \ldots \); however, \( a \neq 4 \) or 8 because \( 11 \mid \sigma(3^4), 11 \equiv 3 \) (mod 8), \( 13 \mid \sigma(3^8) \) and \( 13 \equiv 5 \) (mod 8). Q.E.D.

**Proposition 4.** If \( N \) is OAP, \( \omega(N) \geq 6 \).

**Proof.** Suppose \( N = \Pi_{i=1}^r p_i^{a_i} \) is OAP. Then by Lemma 7 \( pN \) is primitive abundant for some \( p \mid N \). If \( 3 \nmid N, \omega(N) \geq 7 \), for, otherwise,

\[
2 < S(pN) < \prod_{i=1}^r \frac{p_i}{p_i - 1} \leq \frac{5}{4} \frac{7}{6} \frac{11}{10} \frac{13}{12} \frac{17}{16} \frac{19}{18} < 2.
\]

Suppose \( 3 \mid N \). Then \( 3^{12} \mid pN \) by Lemma 8. According to the table of odd primitive abundant numbers \( M \) with fewer than five distinct prime factors in [4] \( 3^{12} \nmid M \).
Hence \( \omega(N) \geq 5 \), and \( N \geq 3^{12}5^27^211^213^2 > 10^{13} \). Then \( 2 > \sigma(N) = 2 - 1/N > 2 - 10^{-13} \), and by Lemma 6 \( \omega(N) \geq 6 \). Q.E.D.

For other results on QP and OAP see [3], [5], [6], [7] and [8].

Computer time for Tables 1 and 2 was over four hours.

**Table 1**

\[ N = \prod_{i=1}^{S} p_i^{a_i} \text{ for which } 2 - 10^{-12} < \sigma(N)/N < 2 + 10^{-12} \]

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**Note:**
(a) If \( b_i = a(p_i) \) and \( c > 0 \), \( Np_i^c \) also satisfies (1). See Lemma 1(a).
(b) See Lemma 1(b). (c) See Lemma 1(c). (d) See Lemma 1(d).
(e) \( 36549767 < p_5 < 36551083 \).


