Computer-Generated van der Waerden Partitions

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Abstract. An exhaustive search made on a PDP-8 mini-computer shows the value of the van der Waerden number $n(2, 5)$ to be 178.

Let $S_n$ denote the set of natural numbers less than or equal to $n$. Van der Waerden's Theorem [2] may be stated as follows: For given natural numbers $k$ and $l$, there exists a natural number $n = n(k, l)$ such that for any $f: S_n \rightarrow S_k$, there is an arithmetic progression of length $l$ of arguments in $S_n$ on which $f$ is constant. In this note we compute the least positive $n = n(k, l)$ for some values of $k$ and $l$.

The proof given in [2] has much overkill and gives an upper bound for $n(k, l)$ that is theoretically constructive but for all practical purposes only existential.

For given $i$, $k$, and $l$, we define $f: S_i \rightarrow S_k$ to be valid if there is no arithmetic progression of the elements in $S_i$ of length $l$ on which $f$ is constant. Consider the following algorithm (for fixed $k$ and $l$) which is designed to find the largest value of $i$ for which $f: S_i \rightarrow S_k$ is valid. (Thus after termination, $i_{\text{max}} + 1$ gives the smallest value of $n(k, l)$):

STEP 1. Set $i = 1$, $f(1) = 1$, $i_{\text{max}} = 1$.
STEP 2. Increment $i$ by 1 and then set $f(i) = 1$.
STEP 3. If $f: S_i \rightarrow S_k$ is valid, then go to STEP 2.
STEP 4. If $f(i) = k$, then go to STEP 6.
STEP 5. Increment $f(i)$ by 1 and then go to STEP 3.
STEP 6. If $i - 1 > i_{\text{max}}$, then set $i_{\text{max}} = i - 1$.
STEP 7. Decrement $i$ by 1.
STEP 8. If $i > 1$, then go to STEP 4.
STEP 9. Stop.

This algorithm was programmed on a PDP-8 mini-computer and run with $k = 2$ and various values of $l$. The resulting values of $i_{\text{max}}$ at termination and approximate running times are shown in the following table:

<table>
<thead>
<tr>
<th>$l$</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$i_{\text{max}}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>time</td>
<td>instantaneous</td>
<td>3 seconds</td>
<td>210 hours</td>
</tr>
</tbody>
</table>

What is perhaps somewhat more interesting is the nature of the valid functions $f: S_{i_{\text{max}}} \rightarrow S_2$. For purposes of illustration we use the two symbols 0 and 1 (rather than 1 and 2) and indicate the functions merely by strings of 0's and 1's. We
indicate only the essentially different strings that were found. All others can be ob-
tained by relabeling (i.e., by interchanging the 0’s and 1’s).

In the case \( l = 3 \), there are three strings: 00110011, 01011010, and 01100110.

In the case \( l = 4 \), there is a repeating subsequence of length 10, which we de-
note by \( X = 0100011101 \). Then all strings are of the form \(*X*X*X*\), where the
*’s (called freebies) denote single entries whose only restriction is that there be no
arithmetic string of four freebies all alike. There are 14 valid strings of length 34.

In the case \( l = 5 \), the repeating subsequence structure is “two deep.” More
precisely, let \( X = 0010000100 \) and \( Y = 0111010001 \). Let \( X’ \) and \( Y’ \) be obtained
from \( X \) and \( Y \), respectively, by relabeling. Then the subsequence \( Z \) is \( X*Y*X’*Y’ \)
(as before, the *’s denote freebies). The valid strings of length 177 are all of the
form \(*Z*Z*Z*Z* \) or reversed from this (end for end). Note that by virtue of the
symmetry exhibited by \( Z \), we may cyclically “conjugate” and obtain, for example,
a reversal by instead setting \( Z = Y*X’*Y*X \). These conclusions about the subse-
quence structure were arrived at in the following way:

The initial string 00010000100 was given (i.e., first freebie set to 0 and next
10 symbols set to \( X \)) to the computer. Sufficient output was printed to establish
that \( Z \) must be constructed and repeated as given above in order to obtain a valid
string of length 177. Then, noting that there are 17 freebies, a count was made of
all valid strings of length 17 with first symbol 0, yielding 24,203. This compares ex-
actly with a count made of all valid strings of length 177 with initial string
00010000100.

A check via computer output also showed that the first 11 symbols must be ini-
tialized to \(*U \) (\( U \) is one of the four “blocks” \( X, Y, X’, \) or \( Y’ \) shown above), and
that the appropriate “conjugate” of \( Z \) must be constructed and repeated in order to
obtain valid strings of length 177.

In conclusion, we note that \( n(2, 5) = 178 \) found above extends the table on
page 31 prepared by V. Chvatal [1]. We would also like to record our agreement
with the value \( n(3, 3) = 27 \) found in this reference.

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1. V. CHVATAL, Some Unknown van der Waerden Numbers, Symposium in Combinatori-
al Methods, Calgary, Alberta, Canada, 1969, pp. 31–33.
2. A. Y. KHINCHIN, Three Pearls of Number Theory, Graylock Press, Rochester, New
York, 1952.