Some Primes with Interesting Digit Patterns

By H. C. Williams

Abstract. Several tables of prime numbers whose forms are generalizations of the form
\((10^n - 1)/9\) of the repunit numbers are presented. The repunit number \((10^{317} - 1)/9\)
is shown to be a prime.

1. Introduction. Considerable interest has been expressed over a long period of
time in the repunit numbers. These are numbers which in decimal notation are made up
only of the unit digit, i.e. numbers of the form \((10^n - 1)/9\). Surveys of the literature
are given in Yates [8], [9]. In this paper we consider some integers whose forms are
generalizations of that of a repunit number and we tabulate some primes of these vari-
ous forms. We also present a new repunit prime.

2. The Extended Forms. The numbers which we discuss in this paper have the
following forms:

\[ N_1(n, r) = (10^n + 1 + 9r - 10)/9, \]
\[ N_2(n, r) = ((9r + 1)10^n - 1)/9, \]
\[ N_3(n, r) = (102^n + 1 + 9(r - 1)10^n - 1)/9, \]
\[ N_4(n, k, r) = 10^r B_k (10^{nk} - 1)/(10^k - 1) + B_r \quad (1 \leq r \leq k), \]

where \(n \geq 1, 1 \leq r \leq 9,\) and

\[ B_k = (10^{k+1} - 9k - 10)/81 \quad (1 \leq k \leq 9). \]

Note that each of these is a generalization of the repunit numbers \((r = 1\) and
\(k = 1)\), although we do not consider these numbers, themselves, here. We illustrate the
digital pattern for \(N_1, N_2, N_3, N_4\) below:

\[ N_1(n, r) = \underbrace{111 \cdots 1}_{n \text{ ones}} r, \]
\[ N_2(n, r) = \underbrace{r111 \cdots 1}_{n \text{ ones}}, \]
\[ N_3(n, r) = \underbrace{111 \cdots 1}_{n \text{ ones}} r \underbrace{111 \cdots 1}_{n \text{ ones}}, \]
\[ N_4(n, r, k) = B_k B_k B_k \cdots B_k B_r, \underbrace{n \text{ \(B_k\)'s}}_{n \text{ \(B_k\)'s}}. \]
where $B_k (1 \leq k \leq 9)$ is a block of digits with the form

$$123 \cdots k.$$ 

For example,

$$N_1(2, 3) = 113, \quad N_2(2, 3) = 311, \quad N_3(2, 3) = 11311,$$

$$N_4(2, 5, 3) = 1234512345123.$$ 

Note that $N_1(n, r)$ cannot be a prime for $r = 2, 4, 5, 6, 8$. Also, $N_3(n, 2)$ cannot be a prime, as

$$N_3(n, 2) = (10^n + 1) (10^{n+1} - 1)/9.$$ 

3. The Tables. Primes of the form $N_1, N_2, N_3, N_4$, which have no more than 100 digits were found. This was done on an IBM system 370 model 168 computer, using the prime testing routines described in Brillhart, Lehmer, and Selfridge [2], Williams and Judd [5], [6].

Naturally these routines were used only on those numbers $N_1, N_2, N_3, N_4$, which have no small prime ($< 1000$) factors. The results of these computations are presented in Tables 1, 2, 3 and 4.

Primes of the form $N_1(n, r)$ ($n \leq 99$) are given in Table 1.

<table>
<thead>
<tr>
<th>$r$</th>
<th>$n \leq 99$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>1, 2, 4, 8, 10, 23</td>
</tr>
<tr>
<td>7</td>
<td>1, 3, 4, 7, 22, 28, 39</td>
</tr>
<tr>
<td>9</td>
<td>1, 4, 5, 7, 16, 49</td>
</tr>
</tbody>
</table>

**Table 1**

In Table 2 we give all those primes of the form $N_2(n, r)$ such that $n \leq 99$.

<table>
<thead>
<tr>
<th>$r$</th>
<th>$n \leq 99$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2, 3, 12, 18, 23, 57</td>
</tr>
<tr>
<td>3</td>
<td>1, 2, 5, 10, 11, 13, 34, 47, 52, 77, 88</td>
</tr>
<tr>
<td>4</td>
<td>1, 3, 13, 25, 72</td>
</tr>
<tr>
<td>5</td>
<td>5, 12, 15, 84</td>
</tr>
<tr>
<td>6</td>
<td>1, 5, 7, 25, 31</td>
</tr>
<tr>
<td>7</td>
<td>1, 7, 55</td>
</tr>
<tr>
<td>8</td>
<td>2, 3, 26</td>
</tr>
<tr>
<td>9</td>
<td>2, 5, 20, 41, 47, 92</td>
</tr>
</tbody>
</table>

**Table 2**
In Table 3 we give all those primes of the form \( N_3(n, r) \) such that \( n \leq 50 \).

<table>
<thead>
<tr>
<th>( r )</th>
<th>( n \leq 50 )</th>
<th>( r )</th>
<th>( n \leq 50 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>1, 2, 19</td>
<td>7</td>
<td>3, 33</td>
</tr>
<tr>
<td>4</td>
<td>2, 3, 32, 45</td>
<td>8</td>
<td>1, 4, 6, 7</td>
</tr>
<tr>
<td>5</td>
<td>1, 7, 45</td>
<td>9</td>
<td>1, 4, 26</td>
</tr>
<tr>
<td>6</td>
<td>10, 14, 40</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Table 3**

Finally, in Table 4, we give all the primes of the form \( N_4(n, k, r) \) for \( m = nk + r \leq 100 \).

<table>
<thead>
<tr>
<th>( k )</th>
<th>( m = nk + r \leq 100 )</th>
<th>( k )</th>
<th>( m \leq 100 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>7, 11, 43</td>
<td>6</td>
<td>Nil</td>
</tr>
<tr>
<td>3</td>
<td>4, 7, 52</td>
<td>7</td>
<td>Nil</td>
</tr>
<tr>
<td>4</td>
<td>55, 71</td>
<td>8</td>
<td>95</td>
</tr>
<tr>
<td>5</td>
<td>21</td>
<td>9</td>
<td>10, 28, 70</td>
</tr>
</tbody>
</table>

**Table 4**

**Remarks.** (1) The number \( N = N_1(83, 3) \) is a base 13 pseudoprime but has not yet been shown a prime. Using the notation of [6] and a factor bound \( B = 5988377680 \), D. H. Lehmer found that \( F_1 = 2^3 \cdot 1531, F_2 = 2 \cdot 3, F_4 = 2 \cdot 5 \cdot 2069 \cdot 215789, F_3 = 7 \cdot 14869, F_6 = 3 \cdot 271 \). All the cofactors are composite. Even using the extended methods of Williams and Holte [7], this is still not enough information to demonstrate the primality of \( N \).

(2) The prime \( N_4(1, 9, 1) \) had been previously discovered by Madachy [4] and the prime \( N_4(3, 9, 1) \) by Finkelstein and Leybourne [3].

4. **A New Repunit Prime.** Because of the interest in repunit primes, all integers of the form \( R_n = (10^n - 1)/9 \) \( n \leq 1000, n \) prime) not divisible by a small prime were tested for pseudoprimality. It was discovered that \( R_n \) is a base 13 pseudoprime only for \( n = 2, 19, 23 \) and 317. It is well known that \( R_2, R_{19} \) and \( R_{23} \) are primes but the result for \( n = 317 \) was surprising, especially in view of the work of Brillhart and Selfridge [1]. In [1] it is stated that \( R_n \) is not a base 3 pseudoprime for any prime \( n \) such that \( 29 \leq n \leq 359 \). \( R_{317} \) was tested again for pseudoprimality with 3 different programs and 50 different prime bases. In each case \( R_{317} \) turned out to be a pseudoprime.

The number \( R_{317} \) is now known to be a prime. We give below the method which was used to demonstrate this. The algorithm used is essentially that of [2].

We first note that


\[ R_{317} - 1 = \frac{(10^{317} - 10)}{9} \]

\[ = 10 \cdot \frac{(10^{79} - 1)}{9} \cdot (10^{79} + 1) \cdot (10^{158} + 1). \]

All the prime factors of \((10^{79} - 1)/9\) which are less than \(10^8\) are 317, 6163, 10271, 307627 and the cofactor is composite. Also, all the prime factors of \(10^{158} + 1\) which are less than 6000 are 101 and 5689. Finally, all the prime factors of \(10^{79} + 1\) which are less than \(10^8\) are 11, 1423 and the cofactor is composite. Fortunately, John Brillhart was able to supply the additional factor 9615060929* of the cofactor; and the resulting cofactor after division by this prime is the 65 digit number

\[ M = 66443174541490579097997510158021076958392938976011506949065646573. \]

This number was found to be a base 13 pseudoprime and was, subsequently, proved prime by using the methods of [5] with a factor bound of \(2 \times 10^6\) and the data

\[
\begin{align*}
F_1 &= 2 \cdot 79, \\
F_2 &= 2 \cdot 3 \cdot 61 \cdot 157 \cdot 199^2, \\
F_4 &= 2 \cdot 5 \cdot 29 \cdot 149 \cdot 421 \cdot 541 \cdot 2137, \\
F_3 &= 19 \cdot 2671 \cdot 2719, \\
F_6 &= 2 \cdot 3 \cdot 13 \cdot 1411783. \quad \text{**}
\end{align*}
\]

Using all the prime factors of \(R_{317} - 1\) found above, we have a completely factored part \(F\) of \(R_{317} - 1\) which exceeds \(3.54 \times 10^{101}\). The tests of [2] were used with a factor bound of \(10^8\) on \(10^{79} - 1\) and 6000 on \(10^{158} + 1\). These tests were satisfied and it follows that if \(p\) is any prime divisor of \(R_{317}\), then

\[ p \equiv 1 \pmod{q_1 q_2 F}, \]

where \(q_1, q_2\) are prime divisors of the cofactors of \(10^{79} - 1\) and \(10^{158} + 1\), respectively. Hence, \(p > 2.12 \times 10^{113}\) and, if \(R_{317}\) is composite, it must be the product of only two primes \(p_1, p_2\) with

\[ p_1 = m_1 F + 1, \quad p_2 = m_2 F + 1. \]

If this is the case, we have

\[ R_{317} - 1 = p_1 p_2 - 1 = m_1 m_2 F^2 + (m_1 + m_2) F; \]

thus,

\[ m_1 + m_2 \equiv (R_{317} - 1)/F \pmod{F}. \]

*This factor was found by S. Wagstaff.

**D. H. Lehmer has found, using the ILLIAC IV, that with the factor bound increased to \(4.2 \times 10^9\) only \(F_2\) increases. It becomes

\[ 2 \cdot 3 \cdot 61 \cdot 157 \cdot 199^2 \cdot 20373173 \cdot 2220165587 \cdot 2746999987. \]

This by itself is sufficient, using only the methods of [2], to prove \(M\) a prime.

He also found, independently, Wagstaff's factor of \(10^{79} + 1\) by searching for prime factors of \(10^{79} - 1, 10^{79} + 1\) and \(10^{158} + 1\) to a factor bound of \(3 \times 10^{11}\). In spite of the fact that the cofactors of \(10^{79} - 1\) and \(10^{158} + 1\) are both composite, no factors were found other than those given above.
Now \((R_{317} - 1)/F \equiv X \pmod{F}\), where \(X > 3.53 \times 10^{101}\) and \(X < F\) and one of \(m_1\) or \(m_2\) must exceed \(\frac{1}{2}X\). Further

\[ R_{317} > m_1 m_2 F^2 > \frac{1}{2} q_1 q_2 X F^2 > 1.32 \times 10^{316} > R_{317}. \]

As this is impossible, \(R_{317}\) can only be a prime.

5. Acknowledgements. The author gratefully acknowledges the extensive amount of computing done by D. H. Lehmer on the ILLIAC IV. He also thanks J. L. Selfridge for making available an unpublished table of factors of \(b^n - 1\) and \(b^n + 1\) for \(b < 13\). This table was put together as a result of work by J. Brillhart, D. H. and Emma Lehmer, J. L. Selfridge, Bryant Tuckerman and S. Wagstaff.

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