A Proof of Convergence and an Error Bound for the Method of Bisection in $\mathbb{R}^n$

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Abstract. Let $S = (X_0, \ldots, X_m)$ be an $m$-simplex in $\mathbb{R}^n$. We define "bisection" of $S$ as follows. We find the longest edge $\langle X_i, X_j \rangle$ of $S$, calculate its midpoint $M = (X_i + X_j)/2$, and define two new $m$-simplexes $S_1$ and $S_2$ by replacing $X_i$ by $M$ or $X_j$ by $M$.

Suppose we bisect $S_1$ and $S_2$, and continue the process for $p$ iterations. It is shown that the diameters of the resulting simplexes are no greater then $(\sqrt{3}/2)^{\lfloor p/m \rfloor}$ times the diameter of the original simplex, where $\lfloor p/m \rfloor$ is the largest integer less than or equal to $p/m$.

1. Introduction and Summary. Recently devised methods for computing roots of a continuous map defined on a simplex (generalized triangle) in $\mathbb{R}^n$ involve a technique of subdivision termed "generalized bisection" ([7], [3], [4]), in which two new simplexes of comparable diameters are formed from the original simplex. An as yet unanswered question concerning such generalized bisections has been: How fast do the diameters of the resulting simplexes tend to zero, as repeated bisection is performed?

In this paper we first define bisection of an $m$-simplex in $\mathbb{R}^n$ and clarify the problem of convergence of the resulting method of bisection. We then prove that, after $p$ repeated bisections, the diameters of the resulting $m$-simplexes in $\mathbb{R}^n$ are no greater than $(\sqrt{3}/2)^{\lfloor p/m \rfloor}$ times the diameter of the original simplex, where $\lfloor p/m \rfloor$ is the largest integer less than $p/m$.


2.1 Definition. Suppose $X_0, \ldots, X_m$ are any $m + 1$ points in $\mathbb{R}^n$ ($1 \leq m \leq n$) and suppose that $\langle X_i - X_0 \rangle_{i=1}^m$ is a linearly independent set of vectors in $\mathbb{R}^n$. Then the closed convex hull of $X_0, \ldots, X_m$, denoted $S = (X_0, \ldots, X_m)$ is called an $m$-simplex in $\mathbb{R}^n$, while the points $X_0, \ldots, X_m$ are called the vertices of $S$ ([1], [2], [3], [4], etc.).

For example, a 3-simplex in $\mathbb{R}^3$ is a tetrahedron, a 2-simplex in $\mathbb{R}^2$ is a triangle, while a 1-simplex in $\mathbb{R}^n$ is a line segment in $\mathbb{R}^n$.

2.2 Remark. The order in which the points $\{X_i\}_{i=0}^m$ are written in the lists $\langle X_0, \ldots, X_m \rangle$ for $S$ determines an orientation of $S$ ([1], [2], [3], etc.). However, the actual closed convex hull is independent of that order; for this reason, for results in this paper we may permute the vertices of $S$.

2.3 Definition. If $S = (X_0, \ldots, X_m)$ is an $m$-simplex in $\mathbb{R}^n$, then we will call each 1-simplex $\langle X_i, X_j \rangle$, $0 \leq i < j \leq n$, an edge of $S$.

2.4 Definition. If $S$ is an $m$-simplex in $\mathbb{R}^n$, then the diameter of $S$ is equal to the quantity $\max_{X, Y \in S} \|X - Y\|$.
2.5 Remark. The diameter of the 1-simplex \( \langle A, B \rangle \) is equal to the length \( \|B - A\|_2 \). Furthermore, by convexity, the diameter of an arbitrary \( m \)-simplex \( S \) is equal to the maximum of the lengths of its edges.

Given an \( m \)-simplex \( S = \langle X_0, \ldots, X_m \rangle \), perhaps the lengths of more than one of the \( \binom{m+1}{2} \) edges \( \{\langle X_i, X_j \rangle \}_{0 \leq i < j \leq m} \) of \( S \) are equal to the diameter of \( S \). However, there is a unique such edge \( \langle X_k, X_{k'} \rangle \) if we require that, if the length of \( \langle X_p, X_q \rangle \) is also equal to the diameter of \( S \), then \( k \leq i \) and \( k' \leq j \).

2.6 Definition. The edge \( \langle X_k, X_{k'} \rangle \) described in the preceding paragraph will be called the selected edge of \( S \).

We now present the definition of bisection.

2.7 Definition ([7], [3]). Suppose \( S_0 = \langle X_0, \ldots, X_m \rangle \) is an \( m \)-simplex, \( \langle X_k, X_{k'} \rangle \) is the selected edge of \( S_0 \), and \( A = (X_k + X_{k'})/2 \) is the midpoint of \( \langle X_k, X_{k'} \rangle \). Then two new simplexes

\[
S_1 = \langle X_0, \ldots, X_{k-1}, A, X_{k+1}, \ldots, X_{k'}, \ldots, X_m \rangle
\]

and

\[
S_2 = \langle X_0, \ldots, X_k, \ldots, X_{k'-1}, A, X_{k'+1}, \ldots, X_m \rangle
\]

may be formed such that the interiors of \( S_1 \) and \( S_2 \) are disjoint and \( S_0 = S_1 \cup S_2 \). We call \( S_1 \) the lower simplex from \( S_0 \), and we call \( S_2 \) the upper simplex from \( S_0 \). The process of producing \( S_1 \) and \( S_2 \) is called bisection of \( S_0 \), while the ordered pair \((S_1, S_2)\) is called the bisection of \( S_0 \).

![Figure 2.1](https://www.ams.org/journal-terms-of-use)
Having defined the above concepts, we can present the main theorem.

3. **The Convergence Theorem and Proof.**

3.1 **Theorem.** Let $S_0$ be an $m$-simplex, let $p$ be any positive integer, and let $S_p$ be any $m$-simplex produced after $p$ bisections of $S_0$. Then the diameter of $S_p$ is no more than $(\sqrt{3}/2)^{p/m}$ times the diameter of $S_0$, where $\lfloor p/m \rfloor$ is the largest integer less than or equal to $p/m$.

3.2 **Corollary.** If $S_0$ is an $n$-simplex in $\mathbb{R}^n$, and $S_p$ is any simplex produced after $p$ bisections of $S_0$, then the diameter of $S_p$ is no greater than $(\sqrt{3}/2)^{p/n}$ times the diameter of $S_0$.

3.3 **Proof of Theorem 3.1.** It suffices to show that, if $p = m$, then the diameter of $S_p$ is no greater than $\sqrt{3}/2$ times the diameter of $S_0$, so assume $p = m$. Then there is a sequence of simplexes $S_q$, $q = 1, \ldots, m$, such that $S_1$ is produced from bisection of $S_0$, $S_q$ is produced from bisection of $S_{q-1}$ for $1 < q < m$, and $S_m$ is produced from bisection of $S_{m-1}$. With the sequence $\{S_q\}_{q=0}^m$ so defined, we set $D_q$ equal to the diameter of $S_q = \langle X_0^{(q)}, \ldots, X_m^{(q)} \rangle$, we let $\langle X_k^{(q)}, X_k^{(q)} \rangle$ be the
selected edge of $S_q$, and we set $d_{ij}^{(q)}$ equal to the length of the edge $(X_i^{(q)}, X_j^{(q)})$, for $0 \leq i < j \leq m$.

We may switch the labels of $k_q$ and $k_q'$ if necessary so that $X_k^{(q+1)} = (X_k^{(q)} + X_{k_q}^{(q)})/2$ and $X_i^{(q+1)} = X_i^{(q)}$ for $0 \leq i \leq m$ and $i \neq k_q$, for $q = 0, \ldots, m - 1$.

We will prove Theorem 3.1 in the setting outlined above by showing that at least $m + (m - 1) + \cdots + (m - q + 1)$ of the $m(m + 1)/2$ distinct edges of $S_q$ each have length no greater than $(\sqrt{3}/2)D_0$, for $1 \leq q \leq m$. The proof will proceed by induction on $q$. Lemma 3.4 (infra) is central to the argument.

If $q = 1$, we invoke (i), Lemma 3.4 to get that each of the $m$ lengths

$$d_{0,k_0}^{(1)}, \ldots, d_{k_0-1,k_0}^{(1)}, d_{k_0,k_0+1}^{(1)}, \ldots, d_{k_0,m}^{(1)}$$

is less than $(\sqrt{3}/2)D_0$. Furthermore, unless $D_1 \leq (\sqrt{3}/2)D_0$ (in which case the conclusion of Theorem 3.1 follows by (iv), Lemma 3.4), we have $k_1 \neq k_0$, and the $m - 1$ edges

$$(3.2) \quad [(X_i^{(2)}, X_j^{(2)})_{0\leq i < k_1, i \neq k_0}] \cup [(X_i^{(2)}, X_j^{(2)})_{k_1 < i \leq m, i \neq k_0}]$$

are distinct from the $m$ edges

$$(3.3) \quad [(X_i^{(2)}, X_j^{(2)}) = (X_i^{(1)}, X_j^{(1)})_{0\leq i < k_0}] \cup [(X_i^{(2)}, X_j^{(2)}) = (X_i^{(1)}, X_j^{(1)})_{k_0 < i \leq m}],$$

where the equalities in (3.3) follow from the definition of bisection. Moreover, application of (i) and (iv), Lemma 3.4 shows that each edge in (3.2) also has a length of at most $(\sqrt{3}/2)D_1 \leq (\sqrt{3}/2)D_0$.

To complete the induction we assume that after $q$ bisections ($q < m$), the $m + (m - 1) + \cdots + (m - q + 1)$ distinct edges in the set

$$\left\{ \bigcup_{j=0}^{q-1} [(X_i^{(q)}, X_j^{(q)})_{0\leq i < k_j, i \neq k_0, \ldots, k_{j-1}}] \right\}$$

$$\cup \left\{ \bigcup_{j=0}^{q-1} [(X_i^{(q)}, X_j^{(q)})_{k_j < i \leq m, i \neq k_0, \ldots, k_{j-1}}] \right\},$$

where $k_j \neq k_j$ for $0 \leq i < j \leq q - 1$, each have length no greater than $(\sqrt{3}/2)D_0$.

Then, unless $D_q \leq (\sqrt{3}/2)D_0$ (which would imply the conclusion of Theorem 3.1 by repeated application of (iv), Lemma 3.4), we have

$$(3.5) \quad k_q \notin \{k_0, \ldots, k_{q-1}\}.$$  

Therefore, the $m - q$ edges

$$[(X_i^{(q+1)}, X_j^{(q+1)})_{0\leq i < k_q, i \neq k_0, \ldots, k_{q-1}}]$$

$$\cup [(X_i^{(q+1)}, X_j^{(q+1)})_{k_q < i \leq m, i \neq k_0, \ldots, k_{q-1}}]$$

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are distinct from the \( m + (m - 1) + \cdots + (m - q + 1) \) edges in the set
\[
\left\{ \bigcup_{j=0}^{q-1} \left\{ \langle X_i^{(q+1)}, X_k^{(q+1)} \rangle = \langle X_i^{(q)}, X_k^{(q)} \rangle \rangle \bigcup_{j=0}^{q-1} \right\} \right\}
\]
(3.7)
where the equalities in (3.7) follow from the definition of bisection. Moreover, we apply (i) of Lemma 3.4, with \( S_{q+1} \) replacing \( S_1 \) and \( S_q \) replacing \( S_0 \), then apply (iv), Lemma 3.4 repeatedly to show that each of the edges in (3.6) has length less than
\[
\frac{\sqrt{3}}{2} D_{q-1} \leq \frac{\sqrt{3}}{2} D_{q-2} \leq \cdots \leq \frac{\sqrt{3}}{2} D_1 \leq \frac{\sqrt{3}}{2} D_0.
\]
Furthermore, we apply the induction hypothesis to each of the edges in (3.7) to show that each of those also has length of at most \((\sqrt{3}/2)D_0\).

Therefore, by induction, after \( m \) bisections, there are \( m + (m - 1) + \cdots + 1 = m(m + 1)/2 \) distinct edges in \( S_m \) whose lengths are at most \((\sqrt{3}/2)D_0\). But an \( m \)-simplex has precisely \( m(m + 1)/2 \) distinct edges, so by Remark 2.5, \( D_m \leq (\sqrt{3}/2)D_0 \).

Repeating the above argument \([p/m]\) times with \( S_m \) and \( D_m \) replacing \( S_0 \) and \( D_0 \), respectively, gives \( D_p \leq (\sqrt{3}/2)^{p/m}D_0 \) for arbitrary integers \( p \); this is the conclusion of Theorem 3.1.

3.4 Lemma. Let \( S_0 = \langle X_0, \ldots, X_m \rangle \) be any \( m \)-simplex in \( \mathbb{R}^n \), suppose \( \langle X_k, X_{k'} \rangle \) \( (\langle X_{k'}, X_k \rangle) \) is the selected edge of \( S_0 \), and suppose
\[
S_1 = \left\langle X_0, \ldots, X_{k-1}, \frac{X_k + X_{k'}}{2}, X_{k+1}, \ldots, X_m \right\rangle
\]
is the lower (the upper) simplex from \( S_0 \). Suppose further that \( d_{i,j}^{(0)} \) and \( d_{i,j}^{(1)} \) are defined as in Theorem 3.1, for \( 0 \leq i < j \leq m \); suppose that \( D_0 = d_{k,k}^{(0)}, D_0 = d_{k,k}^{(1)} \) is the diameter of \( S_0 \) and suppose that \( D_1 \) is the diameter of \( S_1 \). Then
(i) \( d_{i,k}^{(1)} \leq (\sqrt{3}/2)D_0 \) and \( d_{k,i}^{(1)} \leq (\sqrt{3}/2)D_0 \) for \( 0 \leq i < k \) and for \( k < i \leq m \), respectively;
(ii) \( d_{i,k}^{(1)} = d_{k,i}^{(0)} \leq (\sqrt{3}/2)D_0 \);
(iii) \( d_{i,j}^{(1)} = d_{j,i}^{(0)} \) for \( 0 \leq i < j \leq m \) and \( i \neq k \);
(iv) \( D_1 \leq D_0 \).

3.5 Proof of Lemma 3.4. We observe that both (ii) and (iii) follow directly from the definition of bisection. Furthermore, (iv) follows from (i), (ii), and (iii), so we need only prove (i).

To prove (i), we set \( X_j = (x_{j,1}, \ldots, x_{j,n}) \) where \( x_{j,p} \in \mathbb{R} \) for \( 1 \leq p \leq n \) and \( 0 \leq j \leq m \). Then
\[
(d_{i,k}^{(1)})^2 = \sum_{p=1}^{n} \left( x_{i,p} - \frac{x_{k,p} + x_{k',p}}{2} \right)^2 \quad \text{for } 0 \leq j < k
\]
(3.8)
and
\[
(d_{k,i}^{(1)})^2 = \sum_{p=1}^{n} \left( x_{i,p} - \frac{x_{k,p} + x_{k',p}}{2} \right)^2 \quad \text{for } k < j \leq m.
\]
(3.9)
But, for $1 < p < n$ we have
\[
\left( x_{j,p} - \frac{x_{k,p} + x_{k',p}}{2} \right)^2 = x_{j,p}^2 - x_{j,p} x_{k,p} - x_{j,p} x_{k',p} + \frac{x_{k,p}^2}{4} + \frac{x_{k',p}^2}{2} + \frac{x_{k',p}^2}{4}
\]
\[
- \frac{x_{k,p} x_{k',p}}{2} - \frac{x_{k',p}^2}{2} + \frac{x_{k',p}^2}{4}
\]
\[(3.10)\]
\[
= \left( \frac{x_{j,p}^2}{2} - x_{j,p} x_{k,p} + \frac{x_{k,p}^2}{2} \right) + \left( \frac{x_{j,p}^2}{2} - x_{j,p} x_{k',p} + \frac{x_{k',p}^2}{2} \right)
\]
\[
- \left( \frac{x_{k,p}^2}{4} - \frac{x_{k,p} x_{k',p}}{2} + \frac{x_{k',p}^2}{4} \right)
\]
\[
\frac{(x_{j,p} - x_{k,p})^2}{2} - \frac{(x_{k,p} - x_{k',p})^2}{4} + \frac{(x_{j,p} - x_{k',p})^2}{2}.
\]
Hence, summing (3.10) over all $p$ gives
\[
(d_{k,i}^{(1)})^2 = \left[ \sum_{p=1}^{n} (x_{i,p} - x_{k,p})^2 \right] / 2 + \left[ \sum_{p=1}^{n} (x_{i,p} - x_{k',p})^2 \right] / 2
\]
\[
- \left[ \sum_{p=1}^{n} (x_{k,p} - x_{k',p})^2 \right] / 4
\]
\[(3.11)\]
\[
= [(d_{i,j}^{(0)})^2 + (d_{i,k}^{(0)})^2] / 2 - (d_{k,j}^{(0)})^2 / 4,
\]
for $k < j < m$ (we have the same inequality with $d_{i,j}^{(1)}$ replacing $d_{i,k}^{(1)}$ for $0 < j < k$).
However, by assumption we have
\[(d_{i,j}^{(0)})^2 = D_0^2\]
and
\[(d_{i,k}^{(0)})^2 \leq D_0^2, \quad 0 < j < k; \quad (d_{k,j}^{(0)})^2 \leq D_0^2, \quad k < j < m;\]
\[(3.12)\]
\[(d_{i,j}^{(0)})^2 \leq D_0^2, \quad 0 < j < k; \quad (d_{k,j}^{(0)})^2 \leq D_0^2, \quad k' < j < m.\]
\[(3.13)\]
Combining (3.11), (3.12), and (3.13) gives
\[
(d_{k,i}^{(1)})^2 \leq \frac{(D_0^2 + D_0^2)}{2} - \frac{D_0^2}{4} = \frac{3}{4} D_0^2.
\]
Taking square roots of both sides gives
\[
d_{k,i}^{(1)} \leq \frac{\sqrt{3}}{2} D_0, \quad k < j < m.
\]
\[(3.14)\]
To complete the proof of (i), Lemma 3.4, we observe that (3.15) holds when we replace the left member by $d_{i,k}^{(1)}$ for $0 < j < k$.  

It can be seen that the bound given in Theorem 3.1 is sharp for \( p = m \). To show this we set \( m = n - 1 \) and let \( S \) be the \((n-1)\)-simplex whose \( i \)th vertex is the \( i \)th coordinate vector in \( \mathbb{R}^n \). However, numerical experiments verify that, for the same simplex \( S \), the diameters are reduced by a factor of 2 every \( m \) iterations for \( p > m \).

An important related problem is to determine a nonzero lower bound on the ratio of the lengths of the smallest edges of simplexes to the lengths of the largest edges, as bisections are performed. Since the area of each \( m \)-simplex is reduced by a factor of 2 by a simple bisection, such a bound may give a better estimate of the rate of convergence. This problem has been solved for triangles \((n = m = 2)\) \cite{5}.

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