The \( \nu \)-Zeros of \( J_{-\nu}(x) \)

By S. Conde and S. L. Kalla

Abstract. We compute the positive \( \nu \)-zeros of \( J_{-\nu}(x) \), regarded as a function of \( \nu \). 
\( J_\nu(x) \) stands for the Bessel function of first kind of order \( \nu \). Some related inequalities are verified and algorithms the computer uses are described briefly.

1. Introduction. In a recent paper D. Naylor [5] has considered an integral transform adapted to the solution of certain boundary value problems connected with the Helmholtz equation in cylindrical or spherical polar coordinates, where the radial variable \( r \) varies over some infinite interval \( 0 \leq r < \infty \). At infinity a radiation type boundary condition is imposed.

Precisely, suppose that \( f(r) \) is twice continuously differentiable for \( r \geq a \),

\[
r^{-\frac{1}{2}}(rf_r + f_r + k^2 rf) \in L(a, \infty), \quad \lim_{r \to \infty} r^{\frac{1}{2}}f(r)e^{-ikr} \exists \quad \text{and} \quad \lim_{r \to \infty} r^{\frac{1}{2}}(f_r - ikf) = 0,
\]

where \( k \) is real and positive. Let

\[
G(u) = \int_a^\infty \left[ J_u(\kappa r)H_u^{(1)}(\kappa a) - J_u(\kappa a)H_u^{(1)}(\kappa r) \right] f(r) \frac{dr}{r},
\]

where \( a > 0, k > 0 \); then

\[
f(r) = \frac{1}{2} \left[ \int_L \frac{uJ_{-u}(\kappa r)G(u)du}{J_{-u}(\kappa a)} + i\pi \sum_{u=u_n'} \frac{uJ_{-u}(\kappa r)G(u)}{\partial J_{-u}(\kappa a)} \right],
\]

where \( L \) denotes the imaginary axis of the complex \( u \)-plane and the summation is extended over all the positive zeros \( u_n' \) of the function \( J_{-u}(\kappa a) \), regarded as a function of \( u \).

Thus, to apply this interesting transform we need the positive zeros \( u_n' \) of the function \( J_{-u}(\kappa a) \) [3], [5] and [6] regarded as a function of \( u \). As such zeros are not easily available, in this paper we give ten \( \nu \)-zeros (for a particular value of the argument) to eight decimal places.

2. Tables. Tables of numerical values of the \( \nu \)-zeros of \( J_{-\nu}(A) \) are to be found in the microfiche supplement of this issue. The ranges for the parameter \( A \) are

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We consider the applicability of formulas on p. 506 of [6] or see [2], when \( x \) is large \( x > |\nu| \). Thus, large zeros of \( J_{-\nu}(x) \) are

\[
\begin{align*}
A &= 0.001, \quad 0.002, \quad 0.005 \quad (0.005) \quad 0.020, \\
   &\quad 0.030 \quad (0.010) \quad 0.100, \quad 0.120, \quad 0.150 \\
   &\quad 0.200 \quad (0.050) \quad 1.000, \quad 1.100 \quad (0.100) \quad 4.900, \\
   &\quad 5.000 \quad (0.250) \quad 9.750 \quad 10.00 \quad (0.50) \quad 29.50 \\
   &\quad 30.00 \quad (1.00) \quad 990.0, \quad 1000 \quad (5.0) \quad 9500, \\
   &\quad 5000 \quad (200) \quad 4800, \quad 5000 \quad (50) \quad 9500, \\
   &\quad 10^4 \quad (10^3) \quad 9 \times 10^4, \quad 10^5 \quad (2 \times 10^4) \quad 2.8 \times 10^5, \\
   &\quad 3 \times 10^5 \quad (25 \times 10^3) \quad 10^6.
\end{align*}
\]

These results reflect a complete agreement with the numerical values of our tables.

Now,

\[
J_{-\nu}(x) = J_{\nu}(x) \cos \nu \pi - Y_{\nu}(x) \sin \nu \pi.
\]

Hence, zeros of \( J_{-\nu}(x) \) are given by the above formula. This formula is interesting in

\[
(2.1) \quad x \sim \varphi - \frac{4\nu^2 - 1}{8\varphi} - \frac{(4\nu^2 - 1)(28\nu^2 - 31)}{384\varphi^3} - \cdots,
\]

where \( \varphi = (n - (2\nu + 1)/4)\pi \) and \( n \) is any integer.
that the zeros of \( J_{\nu}(x) \) are those of \( J_{\nu}(x) \), when \( \nu \) is a positive integer and the zeros of \( J_{-\nu}(x) \) are those of \( Y_{\nu}(x) \) when \( \nu \) is half an odd integer.

It has been verified that there can be at most one \( \nu \)-zero between two consecutive natural numbers. We observe that for large \( x \), two consecutive \( \nu \)-zeros differ by two, and the large \( \nu \)-zeros are asymptotic to the positive integers [5].

Also, if \( \nu \) is a zero of \( J_{-\nu}(x) \), then from (2.2) it follows that

\[
J_{\nu}(x) = Y_{\nu}(x) \tan \nu \pi.
\]

3. Naylor's Inequalities. Naylor [5] has proved the following inequalities, when \( \nu \) is a zero of \( J_{-\nu}(ka) \),

\[
|J_{\nu}(ka)| J_{-\nu}(kr)| < \sin \frac{\nu \pi}{\nu \pi} \left( \frac{r}{a} \right)^{\nu}, \quad r > a,
\]

and

\[
|J_{\nu}(ka) \frac{\partial}{\partial \nu} J_{-\nu}(ka)| > \frac{\sin \nu \pi}{2\pi} (2/ka)^{2\nu} [\Gamma(\nu)]^2.
\]

Using the \( \nu \)-zeros from our tables, we present here some numerical values for the results, (2.3), (3.1) and (3.2). In the following table the left-hand sides of these results are denoted by \( F_1, G_1 \), and \( H_1 \), where as the right-hand sides are denoted by \( F_2, G_2 \) and \( H_2 \), respectively. Further, \( ka = A \) and \( kr = R \).

<table>
<thead>
<tr>
<th>( A )</th>
<th>( \nu )</th>
<th>( F_1 )</th>
<th>( G_1 )</th>
<th>( H_1 )</th>
<th>( F_2 )</th>
<th>( G_2 )</th>
<th>( H_2 )</th>
</tr>
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<tbody>
<tr>
<td>0.030</td>
<td>0.99977503</td>
<td>0.01501</td>
<td>0.01501</td>
<td>0.00013</td>
<td>0.00030</td>
<td>1.00011</td>
<td>0.00012</td>
</tr>
<tr>
<td>0.120</td>
<td>0.99980605</td>
<td>0.06059</td>
<td>0.06059</td>
<td>0.00090</td>
<td>0.00540</td>
<td>1.00177</td>
<td>0.09115</td>
</tr>
<tr>
<td>1.500</td>
<td>0.95781110</td>
<td>0.48600</td>
<td>0.48690</td>
<td>2.400</td>
<td>0.26138</td>
<td>0.75675</td>
<td>0.57920</td>
</tr>
<tr>
<td>1.800</td>
<td>0.99941460</td>
<td>0.04429</td>
<td>0.04429</td>
<td>2.400</td>
<td>0.00013</td>
<td>0.00014</td>
<td>0.71536</td>
</tr>
<tr>
<td>5.000</td>
<td>2.06830524</td>
<td>0.73977</td>
<td>0.73977</td>
<td>2.400</td>
<td>0.00510</td>
<td>0.00510</td>
<td>0.71536</td>
</tr>
<tr>
<td>7.000</td>
<td>10.9941028</td>
<td>0.08384</td>
<td>0.08384</td>
<td>2.400</td>
<td>0.00510</td>
<td>0.00510</td>
<td>0.71536</td>
</tr>
<tr>
<td>10.000</td>
<td>2.65930622</td>
<td>0.15728</td>
<td>0.15728</td>
<td>30.000</td>
<td>0.01137</td>
<td>0.01137</td>
<td>0.30884</td>
</tr>
<tr>
<td>15.000</td>
<td>15.02145042</td>
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<td>0.18069</td>
<td>45.000</td>
<td>0.00669</td>
<td>0.06690</td>
<td>0.32464</td>
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<tr>
<td>30.000</td>
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<td>0.0145</td>
<td>32.000</td>
<td>0.0145</td>
<td>0.0145</td>
<td>0.06205</td>
</tr>
</tbody>
</table>

As one would expect, \( F_1 = F_2 \), \( G_1 \leq G_2 \), \( R > A \) and \( H_1 \geq H_2 \).

4. Algorithms the Computer Uses. We have used our own routine to calculate \( J_{\nu}(x) \), and then the false position method (Regula falsi or secant method) is used to determine the \( \nu \)-zeros. The function \( J_{\nu}(x) \) is calculated by two different methods, according to its argument.

For \( x < 10 \), the basic series representation

\[
J_{\nu}(x) = \sum_{r=1}^{m} \frac{(-1)^r(x/2)^{\nu+2r}}{\Gamma(r+1)\Gamma(\nu+r+1)}
\]

is used, and the value of \( m \) is chosen in such a way that the truncation error comes out to be of order \( 0.5 \times 10^{-12} \). When \( \nu \) is a negative integer, the relation \( J_{-\nu}(x) = \)
\((-1)^n J_n(x)\) is used to avoid the poles of \(\Gamma(\nu + r + 1)\).

For \(x \gg 10\), we use the following asymptotic formula

\[
J_\nu(x) = \sqrt{\frac{2}{\pi x}} \left[ \cos \left( x - \frac{\nu \pi}{2} - \frac{\pi}{4} \right) \sum_{r=0}^{m} \frac{(-1)^r (\nu, 2r)}{(2x)^{2r}} \right.

\left. - \sin \left( x - \frac{\nu \pi}{2} - \frac{\pi}{4} \right) \sum_{r=0}^{m} \frac{(-1)^r (\nu, 2r + 1)}{(2x)^{2r+1}} \right]
\]

(4.2)

where

\[
(\nu, k) = \frac{(4 \nu^2 - 1)(4 \nu^2 - 3^2) \cdots (4 \nu^2 - (2k - 1)^2)}{2^{2k}k!}
\]

(4.3)

and \(m\) is determined by the relation \(m = 35/\ln|x|\), which leads to a truncation error of order \(10^{-30}\) in both series.

In the calculations \(\nu\) is considered to be a rational number of the form \(u = N/ND\). For this particular case, we set \(ND = 10^8\), and determine \(N\) with a precision of one unit, which guarantees us at precision of \(10^{-8}\) in the tabulation of \(\nu\) in the whole interval \(0 \leq \nu \leq 25\) (that is, up to the first ten positive zeros).

Suppose \(A\) is given. Then by the method of false position \(J_{-\nu}(A)\) is calculated to determine \(\nu\) such that \(J_{-\nu}(A) = 0\). For the given value of \(A\), \(J_{-\nu}(A)\) is evaluated for \(\nu + 0.1 \cdot 10^{-8}\) and \(\nu - 0.1 \cdot 10^{-8}\) and a change in sign for \(J_{-\nu}(A)\) is noted and by linear interpolation the value of \(\nu\) is confirmed.

Uniform asymptotic expansions are useful, if both \(\nu \) and \(x\) are large [4].

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