On Faster Convergence of the Bisection Method for Certain Triangles

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Abstract. Let \( \Delta ABC \) be a triangle with vertices \( A, B \) and \( C \). It is "bisected" as follows: choose \( a \) the longest side (say \( AB \)) of \( \Delta ABC \), let \( D \) be the midpoint of \( AB \), then replace \( \Delta ABC \) by two triangles, \( \Delta ADC \) and \( \Delta DBC \).

Let \( \Delta_{01} \) be a given triangle. Bisect \( \Delta_{01} \) into two triangles \( \Delta_{11}, \Delta_{12} \). Next, bisect each \( \Delta_{1i}, i = 1, 2 \), forming four new triangles \( \Delta_{2i}, i = 1, 2, 3, 4 \). Continue thus, forming an infinite sequence \( T_j, j = 0, 1, 2, \ldots \), of sets of triangles, where \( T_j = \{ \Delta_{2^j i}: 1 \leq i \leq 2^j \} \). It is known that the mesh of \( T_j \) tends to zero as \( j \to \infty \).

It is shown here that if \( \Delta_{01} \) satisfies any of four certain properties, the rate of convergence of the mesh to zero is much faster than that predicted by the general case.

1. Introduction. Let \( \Delta ABC \) be a triangle with vertices \( A, B \) and \( C \). We define the procedure for "bisecting" \( \Delta ABC \) as follows: choose \( a \) the longest side (say \( AB \)) of \( \Delta ABC \), let \( D \) be the midpoint of \( AB \), then divide \( \Delta ABC \) into the two triangles \( \Delta ADC \) and \( \Delta DBC \).

Let \( \Delta_{01} \) be a given triangle. Bisect \( \Delta_{01} \) into two triangles \( \Delta_{11}, \Delta_{12} \). Next, bisect each \( \Delta_{1i}, i = 1, 2 \), forming four new triangles \( \Delta_{2i}, i = 1, 2, 3, 4 \). Set \( T_j = \{ \Delta_{2^j i}: 1 \leq i \leq 2^j \} \), \( j = 0, 1, 2, \ldots \), so \( T_j \) is a set of \( 2^j \) triangles. Define \( m_j \), the mesh of \( T_j \), to be the length of the longest side among the sides of the triangles in \( T_j \).

Clearly \( 0 < m_{j+1} \leq m_j \) for all \( j \geq 0 \). It is shown implicitly in [3] that in fact \( m_j \to 0 \) as \( j \to \infty \). Thus, this bisection method is useful in finite element methods for approximating solutions of differential equations (see e.g. [1]). A modification of such a bisection method can be used in computing the topological degree of a mapping from \( R^3 \) to \( R^3 \) ([4], [5]).

In [2] an explicit bound is obtained for the rate of convergence of \( m_j \): \( m_j \leq (\sqrt{3}/2)^j \sqrt{3}/2 m_0 \), where \( [x] \) denotes the integer part of \( x \). In [2] it is also mentioned that computer experiments indicate that in many cases this bound is unrealistically high; this prompted the present results. We show that if \( \Delta_{01} \) belongs to any one of four sets of equivalence classes of triangles, then we have the substantially improved bounds of Corollaries 1 and 2 below. Much of the notation used is taken from [3].

2. Results.

Definition. Given three positive numbers \( \rho, \sigma, \tau \) such that \( \rho + \sigma + \tau = \pi \), define \( (\rho, \sigma, \tau) \) to be the set of all triangles whose interior angles are \( \rho, \sigma, \tau \).
With this notation we divide the set of all triangles into similarity classes. Note that \((\rho, \sigma, \tau) = (\sigma, \rho, \tau)\) etc.

**Definition.** Given a triangle \(\Delta\), define \(d(\Delta)\), the diameter of \(\Delta\), to be the length of the longest side of \(\Delta\).

**Definition.** Given a similarity class \((\rho, \sigma, \tau)\) choose any triangle \(\Delta \in (\rho, \sigma, \tau)\) and join the midpoint of its longest side to the opposite vertex. The new side ratio \(r\) of \((\rho, \sigma, \tau)\) is defined to be the length of this new side divided by \(d(\Delta)\).

**Remark.** The new side ratio is well-defined since it does not depend on the particular \(\Delta\) chosen in \((\rho, \sigma, \tau)\). By Lemma 5.2(i) of [4] we have \(0 < r < \sqrt{3}/2\) always.

**Definition.** Using the notation of the introduction, an iteration of the bisection method applied to \(\Delta_0\) is defined to be the progression from \(T_j\) to \(T_{j+1}\) for any \(j > 0\). A cycle of the bisection method is defined to be two successive iterations, i.e., the progression from \(T_j\) to \(T_{j+2}\) for any \(j > 0\).

In Figure 1, \(\triangle ABC \in (\rho, \sigma, \tau)\) and we have taken \(\tau < \sigma < \rho\). Thus, \(AC \leq BC \leq AB\) (where 'AC' denotes 'length of AC' etc.). \(D, E, F\) are the midpoints of \(AB, AC, BC\), respectively. Under the additional hypothesis that \(x + \tau > \max \{\sigma, \rho - x\}\) and \(\pi - \rho > \rho - x\), bisections will take place exactly as in Figure 1 (see [3]).

**Notation.** To indicate that \(\Delta_0 \in (\alpha_1, \beta_1, \gamma_1)\) yields \(\Delta_2 \in (\alpha_2, \beta_2, \gamma_2)\) and \(\Delta_3 \in (\alpha_3, \beta_3, \gamma_3)\) when bisectioned, we shall draw a pair of arrows emanating from the triple \((\alpha_1, \beta_1, \gamma_1)\) with one entering each of the triples \((\alpha_2, \beta_2, \gamma_2)\) and \((\alpha_3, \beta_3, \gamma_3)\).

We quote the following result from [3]; it uses the notation of Figure 1.

**Lemma 1 [3, Lemma 4].** If \(\tau < \sigma < \rho, x + \tau > \max \{\sigma, \rho - x\}\) and \(\pi - \rho > \rho - x\), then we must have
\[
(\rho, \sigma, \tau) \longrightarrow (x, \tau, \rho + \sigma - x) \\
\downarrow \\
(\rho - x, \sigma, x + \tau) \longrightarrow (x, \rho - x, \pi - \rho)
\]

**Proof.** See [3].

Thus, after one cycle of the bisection method applied to \(\Delta ABC \in (\rho, \sigma, \tau)\), we have two triangles from \((\rho, \sigma, \tau)\) \((\Delta ADE, \Delta DBF\) above) and two triangles from \((x, \rho - x, \pi - \rho)\) \((\Delta CED, \Delta CFD\) above). Now \(d(\Delta ADE) = d(\Delta DBF) = d(\Delta ABC)/2\). Also,
\[ d(CED) = d(CFD) = CD \text{ (since } \pi - \rho \geq \rho - x \text{ by hypothesis, and } \rho - x \geq x \text{ from [3])}, \]
i.e., \[ d(CED) = d(CFD) = rd(ABC), \] where \( r \) is the new side ratio of \((\rho, \sigma, \tau)\).

Similarly, after one cycle of the bisection method applied to \( \Delta CFD \in (x, \rho - x, \pi - \rho) \), we obtain \( \Delta HJF, \Delta HKF \in (\rho, \sigma, \tau) \) and \( \Delta CJH, \Delta HKD \in (x, \rho - x, \pi - \rho) \).

Here \[ d(CJH) = d(HKD) = d(CFD)/2, \]
and
\[ d(HJF) = d(HKF) = HF = AB/4 = CD/4r = d(CFD)/4r. \]

**Theorem 1.** Assume that \( \tau \leq \sigma \leq \rho, \ x + \tau \geq \max \{\sigma, \rho - x\}, \) and \( \pi - \rho \geq \rho - x. \) Then for \( n \geq 1, \) after \( n \) cycles of the bisection method applied to

(i) \( \Delta \in (\rho, \sigma, \tau) \), we have \( 2^{2n-1} \) triangles in \((\rho, \sigma, \tau)\) each with diameter \( d(\Delta)/2^n \) and \( 2^{2n-1} \) triangles in \((x, \rho - x, \pi - \rho)\) each with diameter \( d(\Delta)/2^n \);  

(ii) \( \Delta' \in (x, \rho - x, \pi - \rho) \), we have \( 2^{2n-1} \) triangles in \((x, \rho - x, \pi - \rho)\) each with diameter \( d(\Delta')/2^n \) and \( 2^{2n-1} \) triangles in \((\rho, \sigma, \tau)\) each with diameter \( d(\Delta')/2^{n+1}. \)

Here \( r \) is the new side ratio of \((\rho, \sigma, \tau)\).

**Proof.** We use induction on \( n \). The case \( n = 1 \) is proven in the remarks following Lemma 1.

Fix \( k > 1. \) Assume that the theorem is true for \( 1 \leq n < k \). We prove it true for \( n = k \).

First, part (i). After one cycle of the bisection method applied to \( \Delta \), we have \( \Delta_1, \Delta_2 \in (\rho, \sigma, \tau) \) with \( d(\Delta_1) = d(\Delta_2) = d(\Delta)/2 \), and \( \Delta_3, \Delta_4 \in (x, \rho - x, \pi - \rho) \) with \( d(\Delta_3) = d(\Delta_4) = rd(\Delta) \). Applying a further \( k - 1 \) cycles to each of these four triangles, we obtain from \( \Delta_1 \) and \( \Delta_2 \) by the inductive hypothesis \( 2^{2k-2} \) triangles in \((\rho, \sigma, \tau)\) each with diameter \( d(\Delta_1)/2^{k-2} = d(\Delta)/2^k \), and \( 2^{2k-2} \) triangles in \((x, \rho - x, \pi - \rho)\) each with diameter \( rd(\Delta_1)/2^{k-2} = rd(\Delta)/2^k \); from \( \Delta_3 \) to \( \Delta_4 \) we get \( 2^{2k-2} \) triangles in \((x, \rho - x, \pi - \rho)\) each with diameter \( d(\Delta_3)/2^{k-2} = rd(\Delta)/2^k \) and \( 2^{2k-2} \) triangles in \((\rho, \sigma, \tau)\) each with diameter \( d(\Delta_3)/r2^k = d(\Delta)/2^k \). Adding totals of identical triangles shows that (i) holds for \( n = k \).

By an analogous argument (ii) holds for \( n = k \). This completes the proof.

**Corollary 1.** Suppose \( \tau \leq \sigma \leq \rho, \ x + \tau \geq \max \{\sigma, \rho - x\}, \) and \( \pi - \rho \geq \rho - x. \) Then in the notation of the introduction

(i) If \( \Delta_0 \in (\rho, \sigma, \tau) \), then \( m_j \leq \max \{r, \frac{1}{2}\} (\frac{1}{2})^{j-1} d(\Delta_0) \) for \( j \geq 1, \) with equality for even \( j \);  

(ii) if \( \Delta_0 \in (x, \rho - x, \pi - \rho) \), then \( m_j \leq \max \{1/2r, 1\} (\frac{1}{2})^{j-1} d(\Delta_0) \) for \( j \geq 1, \) with equality for even \( j. \)

**Proof.** Immediate from Theorem 1.

**Remark.** In practice the conditions of Theorem 1 are more easily checked if expressed in terms of the lengths of sides of triangles. Using the notation of Figure 1,

\( \tau \leq \sigma \leq \rho \) is equivalent to \( AC \leq BC \leq AB, \)
\( x + \tau \geq \max \{\sigma, \rho - x\} \) is equivalent to \( AC \geq \max \{AB/2, CD\}, \)
\( \pi - \rho \geq \rho - x \) is equivalent to \( CD \geq BC/2. \)

Thus, knowing the lengths of \( AC, BC, AB \) and \( CD \) one can immediately decide whether
or not $\Delta A BC \in (\rho, \sigma, \tau)$ satisfies the conditions of Theorem 1. Note that these inequalities and [4, Lemma 5.2(i)] give $1/4 \leq r \leq \sqrt{3}/2$.

Given a triangle such as $\Delta CFD$ with $CD \geq CF > DF$, to decide whether or not $\Delta CFD \in (x, \rho - x, \pi - \rho)$ for some $(\rho, \sigma, \tau)$ where the various angles satisfy the conditions of Theorem 1, bisect $CD$ at $H$ and $CF$ at $J$, then check (as above for $\Delta A BC$) whether or not $\Delta HJF \in (\rho, \sigma, \tau)$ satisfies the conditions of Theorem 1 with $HF \geq FJ \geq JH$.

We now give a theorem similar to Theorem 1 which deals with the other two similarity classes mentioned in Lemma 1.

**Definition.** The smaller sides ratio $s$ of a similarity class $(\rho, \sigma, \tau)$ is obtained by choosing any $\Delta A BC \in (\rho, \sigma, \tau)$ with $AB \geq BC \geq AC$, then setting $s = BC/AC$.

**Theorem 2.** Assume that $\tau < \sigma < \rho, x + \tau > \max\{\sigma, \rho - x\},$ and $\pi - \rho > \rho - x$. Then for $n \geq 1$, after $n$ cycles of the bisection method applied to

(i) $\Delta \in (\rho - x, \sigma, x + \tau)$, we have $2^{n-1}$ triangles in $(\rho - x, \sigma, x + \tau)$ each with diameter $d(\Delta)/2^n$ and $2^{2n-1}$ triangles in $(x, \tau, \rho + \sigma - x)$ each with diameter $sd(\Delta)/2^n$.

(ii) $\Delta' \in (x, \tau, \rho + \sigma - x)$, we have $2^{n-1}$ triangles in $(x, \tau, \rho + \sigma - x)$ each with diameter $d(\Delta')/2^n$ and $2^{2n-1}$ triangles in $(\rho - x, \sigma, x + \tau)$ each with diameter $sd(\Delta')/2^n$.

Here $s$ is the smaller sides ratio of $(\rho, \sigma, \tau)$.

**Proof.** Analogous to that of Theorem 1.

**Corollary 2.** Suppose $\tau < \sigma < \rho, x + \tau > \max\{\sigma, \rho - x\},$ and $\pi - \rho > \rho - x$. Then in the notation of the introduction

(i) if $\Delta_0 \in (\rho - x, \sigma, x + \tau)$, then $m_j \leq s(\frac{1}{2})^{[ij/2]}d(\Delta_0)$ for $j \geq 1$, with equality for even $j$;

(ii) if $\Delta_0 \in (x, \tau, \rho + \sigma - x)$, then $m_j \leq s(\frac{1}{2})^{[ij/2]}d(\Delta_0)$ for $j \geq 1$, with equality for even $j$.

Note that since we are assuming that $AC \geq \max\{AB/2, CD\}$ in Figure 1, we have $1 < s < 2$.

**Remark.** Given a triangle $\Delta RST$ with $RS = d(RST)$, to decide whether or not $\Delta RST \in (\rho - x, \sigma, x + \tau)$ or $(x, \tau, \rho + \sigma - x)$ for some $(\rho, \sigma, \tau)$ where the various angles satisfy the conditions of Theorem 2, bisect $RS$ at $W$ (say). Examine the triangles $\Delta RWT$ and $\Delta WST$. If one of these is in $(\rho, \sigma, \tau)$, where the angles satisfy the conditions of Theorem 2, then

(i) $2WT \geq RS \Rightarrow \Delta RST$ is in the corresponding $(\rho - x, \sigma, x + \tau)$,

(ii) $2WT \leq RS \Rightarrow \Delta RST$ is in the corresponding $(x, \tau, \rho + \sigma - x)$.

Various conditions sufficient for a given triangle $\Delta XYZ$ to lie in one of the four sets of similarity classes considered can be obtained by elementary calculations using the cosine rule for triangles. For example, given $\Delta XYZ \in (\alpha, \beta, \gamma)$ with $XY \geq YZ \geq XZ$ and $\alpha \geq \beta \geq \gamma$, then

(i) if $\cos \gamma < 3/4$, then $\Delta XYZ \in (\rho, \sigma, \tau)$ satisfies the conditions of Theorem 1;

(ii) if $XY/YZ > 2/\sqrt{3}$ and $\cos \gamma < \sqrt{3}/2$, then $\Delta XYZ \in (\rho, \sigma, \tau)$ and $\Delta XYZ \in (x, \rho - x, \pi - \rho)$, both satisfying the conditions of Theorem 1;

(iii) if $3/4 \geq \cos \beta \geq \max\{XZ/XY, XY/XZ\}$, then $\Delta XYZ \in (x, \tau, \rho + \sigma - x)$ satisfies the conditions of Theorem 2.
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