A Legendre Polynomial Integral

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Abstract. Let \( \{ P_n(x) \} \) be the usual Legendre polynomials. The following integral is apparently new.

\[
\int_0^1 P_n(2x - 1) \log \frac{1}{x} \, dx = \frac{(-1)^n}{n(n+1)} \quad \text{for} \quad n \geq 1.
\]

It has an application in the construction of Gauss quadrature formulas on \((0, 1)\) with weight function \(\log (1/x)\).

1. Motivation. For integrals of the type \( \int_a^b f(x)w(x) \, dx \), where \(w(x)\) is positive in \((a, b)\), Gaussian quadrature formulas of the type

\[
\int_a^b f(x)w(x) \, dx \approx \sum_{k=1}^n h_{kn} f(\xi_{kn})
\]

are often useful. The \(\{h_{kn}\}\) and \(\{\xi_{kn}\}\) are chosen to make the formulas exact when \(f(x)\) is a polynomial of degree \(2n - 1\) or less [1]. These formulas are especially useful when \(w(x)\) is singular at one or more points in the interval.

The method of modified moments [2], [3], [4] provides a stable method for calculating the \(\{h_{kn}, \xi_{kn}\}\) if the set of polynomials orthogonal on \((a, b)\) with weight function \(w(x)\) are known. That is, a set of \(\{Q_k\}\), such that

\[
\int_a^b Q_k(x)Q_m(x)w(x) \, dx = 0 \quad \text{if} \quad k \neq m
\]

is desired. Any such family of orthogonal polynomials obeys a three-term recurrence relation [5],

\[
Q_{-1}(x) = 0, \quad Q_0(x) = 1,
\]

\[
xQ_k(x) = a_k Q_{k+1}(x) + b_k Q_k(x) + c_k Q_{k-1}(x), \quad k \geq 1,
\]

with \(a_k \neq 0\).

For some intervals and weight functions, the orthogonal polynomials are known, and there is no problem. For example, if \(a = -1, b = +1, \) and \(w(x) = 1\), the usual Legendre polynomials \(\{P_k(x)\}\) are an orthogonal set,

\[
\int_{-1}^1 P_k(x)P_m(x) \, dx = 0 \quad \text{if} \quad k \neq m.
\]

For most intervals and weight functions, the corresponding orthogonal polynomials are not known. If the moments \(\int_a^b x^k w(x) \, dx\) are known, the \(\{a_k, b_k, c_k\}\) of the unknown set of orthogonal polynomials can be found [2], but the process is nu-
merically unstable \[3\], \[4\]. More generally, if \(\{\overline{Q}_k\}\) is any set of polynomials, not necessarily obeying any orthogonality relation, but obeying a three-term recurrence relation
\[
x\overline{Q}_k(x) = a_k\overline{Q}_{k+1}(x) + b_k\overline{Q}_k(x) + c_k\overline{Q}_{k-1}(x),
\]
the \(\{a_k, b_k, c_k\}\) of the unknown set of orthogonal polynomials can be found \[4\]. For this, the modified moments \(\int_0^1 \overline{Q}_k(x)w(x)\,dx\) are needed. The stability of the process depends on the \(\{\overline{Q}_k\}\). Some particular examples \[3\], \[4\] suggest that, for finite \(a\) and \(b\), the process is probably stable if the \(\{\overline{Q}_k\}\) are themselves orthogonal polynomials with some weight function \(\overline{w}(x)\).

The appropriate orthogonal polynomials for
\[
\int_0^1 f(x) \log \frac{1}{x}\,dx
\]
are not known analytically. The Altran symbolic algebra package \[6\] was used to calculate the modified moments for various sets of orthogonal polynomials. The shifted Legendre polynomials \[5\], \(\{P^*_k(x)\}\), with \(P^*_k(x) = P_k(2x - 1)\), were found to have a particularly simple formula for modified moments, and the algorithm of \[4\] was found to be stable.

### 2. A Legendre Polynomial Integral.

**Theorem.** Let \(P^*_n(x)\) be the \(n\)th shifted Legendre polynomial. Define \(\nu_n = \int_0^1 P^*_n(x)\log(1/x)\,dx\). For \(n > 1\), \(\nu_n = (-1)^n/n(n + 1)\).

**Proof.** By induction. Using \(P^*_k(x) = P_k(2x - 1)\), from \[5\] we obtain
\[
\begin{align*}
P^*_0(x) &= 1, & P^*_1(x) &= 2x - 1, & P^*_2(x) &= 6x^2 - 6x + 1, \\
(k + 1)P^*_{k+1}(x) &= (2k + 1)(2x - 1)P^*_k(x) - kP^*_{k-1}(x), & k &> 2.
\end{align*}
\]

Note that \(P^*_1(1) = 1\). The first three modified moments are \(\nu_0 = 1\), \(\nu_1 = -1/2\) and \(\nu_2 = 1/6\). We define \(\mu_n = \int_0^1 (2x - 1)P^*_n(x)\log(1/x)\,dx\).

Assume \(\nu_k = (-1)^k/k(k + 1)\) for \(k > 2\). Using the recurrence relation,
\[
\nu_{k+1} = \int_0^1 P^*_k(x)\log \frac{1}{x}\,dx = \frac{1}{k + 1} [(2n + 1)\mu_k - k\nu_{k-1}].
\]

Also from \[5\], the derivative of \(P^*_k(x)\) is
\[
\frac{d}{dx}P^*_k(x) = \frac{-k}{2x(1-x)}[(2x-1)P^*_k(x) - P^*_{k-1}(x)].
\]

Integrate by parts in the definition of \(\mu_k\) to obtain
\[
\mu_k = P^*_k(x) \left[ x(1 - x) \ln x + \frac{1}{2} x^2 - x \right]_0^1
+ \frac{k}{4} \int_0^1 \frac{x - 2}{1-x} [(2x - 1)P^*_k(x) - P^*_{k-1}(x)]\,dx.
\]

Simplifying, and using \(P^*_k(1) = 1\),
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\[ \mu_k = -\frac{1}{2} - \frac{k}{2} \mu_k + \frac{k}{2} \nu_{k-1} - \frac{1}{2} \int_0^1 x(x-2) \left[ \frac{d}{dx} P_k^*(x) \right] dx. \]

The last integral may be integrated by parts, giving

\[ -\frac{1}{2} x(x-2) P_k^*(x) \bigg|_0^1 + 2 \int_0^1 (x-1) P_k^*(x) \, dx. \]

The integrated term is 1/2, and the integral is zero for \( k > 1 \) because of the orthogonality of the \( \{P_k^*\} \). Thus,

\[ \mu_k = \frac{k}{2} (\nu_{k-1} - \mu_k), \quad \mu_k = \frac{k}{k+2} \nu_{k-1}. \]

Inserting this result in (1),

\[ \nu_{k+1} = \frac{k}{k+1} \left[ \frac{2k+1}{k+2} - 1 \right] \nu_{k-1} = \frac{k(k-1)}{(k+1)(k+2)} \frac{(-1)^{k-1}}{k(k-1)} \]

\[ = \frac{(-1)^{k+1}}{(k+1)(k+2)}. \]

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