On the Preceding Paper

"A Legendre Polynomial Integral"

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Abstract. The modified moments of the distribution $d\sigma(x) = x^\alpha \ln(1/x) dx$ on $[0, 1]$, with respect to the shifted Legendre polynomials, are explicitly evaluated.

The result in the theorem of Section 2 of [1] can be generalized as follows: Let

$$
\nu_n(\alpha) = \int_0^1 x^\alpha \ln(1/x) P_n^*(x) dx, \quad \alpha > -1, \quad n = 0, 1, 2, \ldots,
$$

where $P_n^*(x) = P_n(2x - 1)$ is the shifted Legendre polynomial of degree $n$. Then

$$
\nu_n(\alpha) = \left\{ \begin{array}{ll}
(-1)^{n-m} \frac{m!^2(n-m-1)!}{(n+m+1)!}, & \alpha = m < n, m \geq 0 \text{ an integer}, \\
\frac{1}{\alpha+1} \left\{ \frac{1}{\alpha+1} + \sum_{k=1}^{n} \left( \frac{1}{\alpha+1+k} - \frac{1}{\alpha+1-k} \right) \right\} \frac{\alpha+1-k}{\alpha+1+k}, & \text{otherwise}.
\end{array} \right.
$$

The result in [1] is the case $\alpha = 0$ of (1). For the proof, we note that

$$
\nu_n(\alpha) = -2^{-(\alpha+1)} \int_0^1 (1+t)^\alpha \ln(\frac{1}{2}(1+t))P_n(t) dt
$$

(2)

$$
= -2^{-(\alpha+1)} \lim_{n \to \infty} \left\{ \int_0^1 (1+t)^\alpha \ln(1+t)P_n(t) dt - \ln 2 \int_0^1 (1+t)^\alpha P_n(t) dt \right\},
$$

where $P_n(t)$ is the Legendre function of degree $\nu$. It is well known [2, p. 316, Eq. (15)] that

$$
\int_0^1 (1+t)^\alpha P_\nu(t) dt = \frac{2^{\alpha+1} \Gamma(\alpha+1)}{\Gamma(\alpha+\nu+2) \Gamma(\alpha+1-\nu)}, \quad \alpha > -1.
$$

(3)

Differentiating (3) with respect to $\alpha$ gives

$$
\int_0^1 (1+t)^\alpha \ln(1+t)P_\nu(t) dt
$$

(4)

$$
= \frac{2^{\alpha+1} \Gamma(\alpha+1)}{\Gamma(\alpha+\nu+2) \Gamma(\alpha+1-\nu)} \left\{ \ln 2 + 2 \psi(\alpha+1) - \psi(\alpha+\nu+2) - \psi(\alpha+1-\nu) \right\},
$$

with $\psi(x) = \Gamma'(x)/\Gamma(x)$ the logarithmic derivative of the gamma function. The assertion (1) now follows by inserting (3) and (4) in (2) and by using the recurrence relations

$$
\Gamma(x+1) = x\Gamma(x), \quad \psi(x+1) = \psi(x) + 1/x,
$$

together with the fact that for any integer

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$r \geq 0,$

$$\frac{\psi(-r + \epsilon)}{\Gamma(-r + \epsilon)} \rightarrow (-1)^{r-1}r! \quad \text{as } \epsilon \rightarrow 0.$$  

The method of proof also allows the evaluation of integrals of the form

$$\nu_{n,k}(\alpha) = \int_0^1 x^\alpha \ln(1/x)^k P_n^*(x) \, dx,$$

by repeatedly differentiating (4) with respect to $\alpha$. 

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