The Hankel Power Sum Matrix Inverse and the Bernoulli Continued Fraction

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Abstract. The $m \times m$ Hankel power sum matrix $W = V V^T$ (where $V$ is the $m \times n$ Vandermonde matrix) has $(i, j)$-entry $S_{i+j-2}(n)$, where $S_p(n) = \frac{n^p}{p!}$, $p = \frac{n^p}{p!}$. In solving a statistical problem on curve fitting it was required to determine $f(m)$ so that for $n > f(m)$ all eigenvalues of $W^{-1}$ would be less than 1. It is proved, after calculating $W^{-1}$ by first factoring $W$ into easily invertible factors, that $f(m) = (13m^2 - 5)/8$ suffices. As by-products of the proof, close approximations are given for the Hilbert determinant, and a convergent continued fraction with $m$th partial denominator $m^{-1} + (m + 1)^{-1}$ is found for the divergent Bernoulli number series $\Sigma B_{2k}(2x)^{2k}$.

1. Introduction. Defined as the product $W = V V^T$ of the $m \times n$ Vandermonde matrix $V = (j^{i-1})$ with its transpose $V^T$, the $m \times m$ Hankel power sum matrix $W = W_m$ has $(i, j)$-entry $S_{i+j-2}(n)$, where

$$S_p = S_p(n) = \sum_{k=1}^{n} k^p = \frac{n^{p+1}}{p+1} + \frac{n^p}{2} + \sum_{k=1}^{[p/2]} \frac{B_{2k}}{2k} \binom{p}{2k-1},$$

and where $B_{2k}$ are the Bernoulli numbers [3], [6]:

$$B_2 = 1/6, \quad B_4 = -1/30, \quad B_6 = 1/42, \quad B_8 = -1/30, \quad B_{10} = 5/66, \ldots .$$

In solving a statistical problem involving the fitting of polynomial curves of degree $m$ to $n > m$ points, for increasing $m$ and $n$, it was required [4] to find a function $f(m)$, such that, whenever $n > f(m)$, the eigenvalues $\mu_k$ of $M = W^{-1}$ would all be less than 1. We evaluate $w_m = \det W_m$ in Section 2 as

$$w_m = \det W_m = h_m \prod_{i,j=1}^{m} (n + i - j),$$

where $h_m$ is the determinant of the Hilbert matrix $H_m$, and we obtain close estimates for $h_m$. In Section 3 we factor $W_m$ as a product of easily invertible matrices of which only diagonal matrices involve $n$, and we also explicitly invert $W_m$ and $H_m$. In Section 4 we estimate the trace of $M = W^{-1}$ and find that the function

$$f(m) = (13m^2 - 5)/8$$

suffices for powers of $M$ to converge when $n > f(m)$.

As a by-product of this investigation, we find in Section 5 that the divergent
asymptotic series $B(x)$ with Bernoulli number coefficients

$$B(x) = \sum_{k=1}^{\infty} B_{2k}(2x)^{2k},$$

related to the Laplace transform of $x \coth x - 1$, has the convergent continued fraction expansion

$$B(x) = \frac{x^2}{1 + \frac{x^2}{1/2 + \frac{x^2}{1/2 + \frac{x^2}{1/3 + \frac{x^2}{1/3 + \frac{x^2}{\ldots}}}}}.$$

In fact, $B(1/12) = \pi^2 - 9.865$ is given with error $< 2 \times 10^{-12}$ by the sixth convergent of this continued fraction.

2. The Determinants. Since $S_0(n) = n$ and $n(n + 1)/2$ divides $S_k(n)$ for $k > 0$, it follows directly that $w_m = \det W_m$ has the algebraic factor $n^m(n + 1)^{m-1}$. For $r < m$, $(n-r)^{m-r}$ is also an algebraic factor of $w_m$, since the matrix $W_m(n)$ has rank $r$ and nullity $m - r$, when $n = r$ for $r = 1, 2, \ldots, m - 1$. Since the polynomials $S_p(n)$ are generated by the function

$$G(x, n) = \frac{(e^{xn} - 1)/(1 - e^{-x})}{\sum_{p=0}^{\infty} S_p(n)x^p/p!},$$

we see from the identity

$$G(x, n) + G(-x, -n - 1) + 1 = 0$$

that

$$S_p(n) + (-1)^p S_p(-n - 1) + \delta_{p,0} = 0.$$

Hence, $w_m$ has the algebraic factor $(n + 1 + r)^{m-1-r}$. We have found $m^2$ linear functions of $n$ as factors of the polynomial $w_m(n)$ which is of degree $m^2$ in $n$. The remaining factor is the determinant of the leading coefficients $1/(i+j-1)$, namely the determinant $h_m$ of the ill-conditioned Hilbert matrix $H_m$ of order $m$ [5], [7]. This proves Eq. (1.3). If we take $m = n$ in (1.3), the Vandermonde matrix $V$ in $W = VVT$ is square. Its determinant $v_m$ is

$$v_m = \det V_m = 1! 2! 3! \cdots (m-1)! \equiv (m-1)!!.$$

Hence by (1.3) and (2.3)

$$h_m = \det V_m^2 \prod_{i,j=1}^{m} (m + i - j) = v_m^4/v_{2m}.$$

The ratio of successive $h_m$'s is

$$h_m/h_{m+1} = (2m + 1)!(2m)!/(m!)^4 = (2m + 1) \left( \frac{2m}{m} \right)^2.$$

The following theorem gives a close approximation for $h_m$.

**Theorem 2.1.** The determinant $h_m$ of the Hilbert matrix $H_m$ of order $m$ is
given to 10 significant figures for $m > 4$ by

\begin{equation}
(2.6) \quad h_m = 4^{-m(m-1)}(\pi/2)^{m-1} e^{-m-1/4} \exp R_m,
\end{equation}

where the remainder function $R_m$ is defined by

\begin{equation}
(2.7) \quad R_m = \int_0^\infty (e^{-2t} - e^{-2mt}) \tanh^2 (t/2)(4t)^{-1} dt
\end{equation}

and is approximated to 9 decimals for $m \geq 5$ by

\begin{equation}
(2.8) \quad R_m = 0.013081539 - 2^{-6} m^{-2} + 2^{-8} m^{-4} - 2^{-8.5} m^{-6}
+ 2^{-8} m^{-8} - 2^{-7} m^{-10}.
\end{equation}

Proof. The ratio $h_m/h_{m+1}$ in (2.5) is related to the Wallis approximation $\pi_m$
for $\pi$ by

\begin{equation}
(2.9) \quad \frac{\pi_m}{2} = \frac{2}{1} \frac{2}{3} \frac{4}{3} \frac{4}{5} \cdots \frac{2m}{2m-1} \frac{2m}{2m+1} = 2^m \pi_{m+1}/\pi_m.
\end{equation}

If we express $\ln n$ in the form

\begin{equation}
(2.10) \quad \ln n = \int_1^n \frac{ds}{s} = \int_0^\infty \int_1^n e^{-st} ds dt = \int_0^\infty (e^{-t} - e^{-nt}) t^{-1} dt,
\end{equation}

then $\ln(\pi_m/2)$ and its limit $\ln(\pi/2)$ are expressible as

\begin{equation}
(2.11) \quad \ln(\pi_m/2) = \int_0^\infty (e^{-t} - 2e^{-2t} + 2e^{-3t} - 2e^{-2mt} + e^{-(2m+1)t}) t^{-1} dt
= \int_0^\infty (e^{-t} - e^{-(2m+1)t})(1-e^{-t})(1+e^{-t})^{-1} t^{-1} dt,
\end{equation}

\begin{equation}
(2.12) \quad \ln(\pi/2) = \int_0^\infty e^{-t} \tanh (t/2) t^{-1} dt,
\end{equation}

\begin{equation}
(2.13) \quad \ln(\pi/\pi_k) = \int_0^\infty e^{-(2k+1)t} \tanh (t/2) t^{-1} dt,
\end{equation}

\begin{equation}
(2.14) \quad (\pi/2)^{m-1} \prod_{k=1}^{m-1} (\pi_k/\pi) = 2^m \pi_{m+1}/\pi_1.
\end{equation}

Summing in (2.13) from $k = 1$ to $m - 1$ yields

\begin{equation}
(2.15) \quad \ln[(\pi/2)^{m-1}2^{-2m}h_m] = \int_0^\infty (e^{-2t} - e^{-2mt})(e^{t/2} + e^{-t/2})^{-2} t^{-1} dt
= (1/4) \ln(2m/2) - R_m
\end{equation}

by (2.10), where $R_m$ is defined by (2.7). Equation (2.6) follows from (2.15). To obtain (2.8) we evaluate $R_4 = 0.011219610988$ from (2.6) setting $h_4 = 1/6048000$ in (2.6). Then we compute $R_m - R_m$ from (2.7) by replacing $\tanh(t/2)$ by the first five terms of its series, and set $m = 4$ to get $R_m$ in (2.8). We check the tenth decimal by working from $h_5$ instead. This gives $\exp R_m$ and $h_m$ accurate to 10 significant figures.
For \( m = 20 \) we find \( R_{20} = .0130425009 \) and

\[
(2.16) \quad h_{20} = 4.206178954 \times 10^{-226}.
\]

The matrices \( H_m \) and \( W_m(n) \) are ill conditioned. In fact, \( W_3(3) \) has the eigenvalues \( \lambda_1 = 113.4132, \lambda_2 = 1.564253, \lambda_3 = 0.2254695 \) and the conditioning ratio \( \lambda_1/\lambda_3 = 5030 \). So the usual computer methods for inverting \( W_m(n) \) are unreliable [5], [7].

3. Inversion by Factoring. To invert the ill-conditioned \( m \times m \) matrix \( W = W_m(n) \) with \((i, j)\)-entry \( S_{i+j-2}(n) \), we first factor it into easily invertible factors, restricting the variable \( n \) to diagonal matrix factors \( EP = (\text{diag } e_i p_i) \) and \( Q = (\text{diag } q_j) \), where

\[
(3.1) \quad e_i = (-1)^{i-1}, \quad p_i = \binom{n + i - 1}{n - m}, \quad q_j = \binom{n - j}{n - m}.
\]

We denote by \( T = (t_{ij}) \) the lower Pascal triangle matrix with

\[
(3.2) \quad t_{ij} = \binom{i - 1}{j - 1} = \binom{i - 1}{i - j} = (-1)^{i+j} \binom{-j}{i - j}.
\]

We note that \( ETE \) has entries \((i-j)\), so

\[
(3.3) \quad (TETE)_{ij} = \sum_{k=j}^{i} \binom{i - 1}{i - k} \binom{-j}{k - j} = \binom{i - j - 1}{i - j} = \delta_{ij}
\]

and \( T^{-1} = ETE \). Next, we define an \( m \times m \) lower triangular row stochastic matrix \( A = (a_{ij}) \) that converts the integral powers in \( V \) into binomial coefficients by the formula

\[
(3.4) \quad (AV)_{ik} = \sum_{r=1}^{i} a_{ir} k^{-1} = \binom{k + i - 2}{i - 1}, \quad k = 1, 2, \ldots, n.
\]

The \( a_{ir} \) are related to Stirling numbers of the first kind, and

\[
(3.5) \quad A = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & \cdots \\
0 & 1 & 0 & 0 & 0 & \cdots \\
0 & 1/2! & 1/2! & 0 & 0 & \cdots \\
0 & 2/3! & 3/3! & 1/3! & 0 & \cdots \\
0 & 6/4! & 11/4! & 6/4! & 1/4! & \cdots 
\end{bmatrix} = (a_{ij}).
\]

From (3.5) and (3.4) we obtain

\[
(3.6) \quad (AV)_{ik} = \binom{i + k - 2}{i - 1} = \sum_{j=1}^{i} \binom{i - 1}{i - j} \binom{k - 1}{j - 1} = \sum_{j=1}^{k} t_{ij} \binom{k - 1}{j - 1}.
\]

To factor \( W \) we now write
HANKEL POWER SUM MATRIX INVERSE

\[(AV(T^{-1}AV)^T)_{ij} = \sum_{k=j}^{n} \binom{k + i - 2}{i - 1} \binom{k}{j - 1},\]

\[(3.7)\]

\[= \binom{i + j - 2}{i - 1} \sum_{k=j}^{n} \binom{k + i - 2}{j + i - 2}.\]

Summing over \(k\) yields

\[(AW(T^{-1}A)^T)_{ij} = \binom{i + j - 2}{i - 1} \binom{n + i - 1}{j + i - 1},\]

\[(3.8)\]

\[= p_i \binom{i + j - 2}{i - 1} \binom{m + i - 1}{j + i - 1}/q_j.\]

**Theorem 3.1.** The inverse matrix \(M = W^{-1}\) has the factorization

\[(3.9)\]

\[M = W^{-1} = A^T E B A, \quad B = T^T Q T T^T T P^{-1},\]

where \(E, P, Q, T, A\) are defined in (3.1), (3.2) and (3.5).

**Proof.** Entries of \(T T^T\) and \(T T^T T\) are

\[(3.10)\]

\[(TT^T)_{ik} = \sum_{r=1}^{i} \binom{i - 1}{i - r} \binom{k - 1}{r - 1} = \binom{i + k - 2}{i - 1},\]

\[(3.11)\]

\[(TT^T T)_{ij} = \sum_{k=j}^{m} \binom{i + k - 2}{k - 1} \binom{k - 1}{j - 1} = \binom{i + j - 2}{i - 1} \sum_{k=j}^{m} \binom{k + i - 2}{j + i - 2}.\]

Summing over \(k\) yields

\[(3.12)\]

\[(TT^T T)_{ij} = \binom{i + j - 2}{i - 1} \binom{m + i - 1}{j + i - 1}.\]

Combining (3.8) and (3.12), we have

\[(3.13)\]

\[A W A^T = P T T^T T Q T T^T T P^{-1}.\]

Since the diagonal sign matrix \(E\) commutes with \(P\) and \(Q\) but transforms \(T\) and \(T T\) into their inverses,

\[(3.14)\]

\[(A W A^T)^{-1} = E T^T Q T T^T T P^{-1} E = E B E\]

and (3.9) is proved.

Equation (3.12) provides a simple method for inverting the Hilbert matrix \(H\).

**Theorem 3.2.** The inverse of the \(m \times m\) Hilbert matrix \(H = (h_{ij})\) with \((i, j)\)-entry \(h_{ij} = 1/(i + j - 1)\) is given by

\[(3.15a)\]

\[(H^{-1})_{ij} = d_i d_j h_{ij} d_i d_j,\]

\[(3.15b)\]

\[d_i' = \binom{-m - 1}{i - 1}, \quad d_j = \binom{m - 1}{j - 1}.\]
Proof. Factoring (3.12) yields
\[(3.16) \quad (ETT^T T)_{ij} = (-1)^{i-j} \binom{m+i-1}{i-1} \frac{m}{i+j-1} \binom{m-1}{j-1} = d_i^j h_j d_j.\]

Since \(ETT^T T\) is involutory, \(D'HD = (D'HD)^{-1}\). Also,
\[(3.17) \quad h_m = \det H_m = \pm 1 \prod_{i=1}^m d_i d_i'.\]

Although the matrix \(B\) in (3.9) is symmetric, its symmetry is not obvious from formula (3.9).

**Theorem 33.** The symmetric matrix \(B = (b_{ij})\) in (3.9) has entries expressible in terms of descending factorials \((x)_r = x(x-1) \cdots (x-r+1)\) as follows:
\[(3.18) \quad b_{ij} = \sum_{r \geq i+j-1} \frac{(m+i-1)_r (m+j-1)_r}{(n-m+r)_r r!} \binom{r-1}{i-1, j-1},\]
where \((r-1)_{i,j-1}\) is the trinomial coefficient
\[
\begin{pmatrix}
    r-1 \\
    i+2 \\
    j-1
\end{pmatrix}
\binom{i+j-2}{i-1}.
\]

Proof. To transform \(B\) in (3.9) we evaluate
\[(3.19) \quad (T^T Q T T^T)_{is} = \sum_{r=i}^m \binom{r-1}{i-1} \binom{n-r}{n-m} \binom{r+s-2}{r-1},\]
\[(3.20) \quad (B P)_{ij} = \sum_{s=j}^m \binom{i+s-2}{i-1} \binom{n+s-1}{m-i} \binom{s-1}{j-1},\]
\[(3.21) \quad b_{ij} = \sum_{r \geq i+j-1} \binom{m+i-1}{r} \binom{r-1}{i-1, j-1} \binom{m+j-1}{n-m+r}.\]

Writing \((x)_r = r!(x)^r\), Eq. (3.21) becomes (3.18).
To conserve space in displaying the symmetric matrices $M_m(n) = W_m^{-1}(n)$ we show the upper half of $M_3$ and the lower half of $M_4$.

(3.22)

<table>
<thead>
<tr>
<th>$M_3$</th>
<th>$M_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$16n^2 + 24n^2 + 56n + 24$</td>
<td>$9(n^2 + n) + 6$</td>
</tr>
<tr>
<td>$-120(n^2 + n) - 100$</td>
<td>$-18(2n + 1)$</td>
</tr>
<tr>
<td>$120(2n + 1)$</td>
<td>$30$</td>
</tr>
<tr>
<td>$120(n^2 + 4.5n^2 + 7n^2 + 5n + 11/6)$</td>
<td>$(n + 3)^2$</td>
</tr>
<tr>
<td>$-300(n + 1)(3n + 2)(3n + 5)$</td>
<td>$360(n + 1)(9n + 13)$</td>
</tr>
<tr>
<td>$-140$</td>
<td>$-4200(n + 1)$</td>
</tr>
<tr>
<td>$-300(n + 1)(3n + 2)(3n + 5)$</td>
<td>$2800(n + 3)^5$</td>
</tr>
<tr>
<td>$280(n^2 + 15n + 11)$</td>
<td>$2800$</td>
</tr>
</tbody>
</table>

4. Estimation of $\text{tr} M$. Since $W$ and $M$ are positive definite for $n > m$, all eigenvalues $\mu_k$ of $M$ will satisfy $\mu_k < 1$ if $\text{tr} M < 1$, $m > 1$. For $A$ in (3.5) and $e_l = (-1)^{l-1}$, the first and last diagonal entries of $M = M_m$ are $b_{11}$ and $b_{mm}/((m - 1)!)^2$. Numerical computation shows that the maximum $n$ for which $\det(W_m(n) - I) = 0$ are given for $m = 1, 2, 3, 4$ by

(4.1) $(m, n) = (1, 1), (2, 5.82090), (3, 13.3776), (4, 24.24453)$.

The parabola through the first three points is

(4.2) $n = g(m) = 1.5679m^2 - .0828m - .4851$,

and we find $g(4) = 24.270 > 24.24453$. A slightly higher value than (4.2) will be required for $\text{tr} M < 1$. We first estimate the dominant diagonal entry $b_{11}$ of $M$.

(4.3) $b_{11} = \sum_{r=1}^{m} \binom{m}{r}^2 \binom{n}{m} \binom{n - m + r}{r} = \sum_{r=1}^{m} \binom{m}{r} \binom{n}{m - r}$

(4.4) $1 + b_{11} = \binom{n + m}{m} / \binom{n}{m} = \prod_{k=1}^{m} \frac{2n + 1 + (2k - 1)}{2n + 1 - (2k - 1)}$

(4.5a) $\ln(1 + b_{11}) = \sum_{k=1}^{m} \ln \left( \frac{1 + (2k - 1)(2n + 1)}{1 - (2k - 1)(2n + 1)} \right) = \sum_{r=1}^{m} \frac{\theta(m, r)}{(2r - 1)(2n + 1)^{2r - 1}}$

where

(4.5b) $\theta(m, r) = \sum_{k=1}^{m} 2(2k - 1)^{2r - 1} < \int_{0}^{2m} x^{2r - 1} dx = (2m)^{2r}/2r$.

We now assume the inequalities $n > f(m)$ in (1.4).

Theorem 4.1. The matrix $M = W_m^{-1}(n)$ has trace $< 1$ if

(4.6) $n > 1.625m^2 - .625$ and $m \geq 5$. 
Proof. If (4.6) is satisfied for \( m = 5 \), then \( n > 40 \), and

\[
(4.7a) \quad b_{11} \leq \frac{45}{(40)} - 1 = 62.639/73.112 = .856754.
\]

If (4.6) is satisfied for \( m > 6 \), then \( \frac{(2n + 1)/2m^2 > 1.6215} \) and

\[
\ln(1 + b_{11}) < \frac{2m^2}{2n + 1} \sum_{r=1}^{\infty} \left( \frac{2m}{2n + 1} \right)^{2r-2} / r(2r - 1)
\]

\[
< \frac{1}{1.6215} \sum_{r=1}^{\infty} \left( \frac{1}{9.729} \right)^{2r-2} / r(2r - 1) < .62777,
\]

\[
(4.7b) \quad b_{11} < .8734 \quad \text{for} \quad m > 6, n > (13m^2 - 5)/8.
\]

The rest of \( \text{tr} \ M \) is given by

\[
(4.9) \quad \text{tr} \ M - b_{11} = \sum_{k=2}^{m} \sum_{i,j,k} a_{ik}(-1)^i b_{ij}(-1)^j a_{jk}.
\]

We replace \( i, j, r \) by \( i + 1, j + 1, r + 2 \) and write

\[
(4.10) \quad b_{i+1,j+1} = \sum_{k=1}^{m-1} y_{ij}^{(r)}, \quad y_{ij}^{(r)} = \frac{(m + 1)(m + 1)_{r+2}(m + 1)_{r+2}}{(m - n + r + 2)_{r+2}(r + 2)!} \left( \begin{array}{c} r+1 \\ i, j \end{array} \right) = y_{ij}^{(r)}.
\]

Then

\[
(4.11) \quad \text{tr} \ M - b_{11} = y_{11}^{(1)} \sum_{r=1}^{2m-1} \varphi_{mn}(r), \quad \varphi_{mn}(r) = \sum_{i+j=1}^{r+1} c_{ij} y_{ij}^{(r)} / y_{11}^{(1)},
\]

where the entries of the \((m - 1) \times (m - 1)\) matrix \( C = (c_{ij}) \) are

\[
(4.12a) \quad c_{ij} = (-1)^{i+j} \sum_{k=1}^{m-1} a_{i+1,k} a_{j+1,k} = c_{ij},
\]

\[
(4.12b) \quad C = \frac{1}{720} \begin{bmatrix}
720 & -360 & 240 & -180 & 144 & \cdots \\
-360 & 360 & -300 & 255 & -222 & \cdots \\
240 & -300 & 280 & -255 & 233 & \cdots \\
-180 & 255 & -255 & 242.5 & -228.5 & \cdots \\
144 & -222 & 233 & -228.5 & 220.1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\]

The dominant term \( y_{11}^{(1)} \) satisfies

\[
(4.13) \quad y_{11}^{(1)} < \frac{(m + 1)_{3}(m + 1)_{3}/3}{(13m^2/8 - m + 19/8)_{3}} < \frac{6363/3}{(38)_{3}} = \frac{200}{2109} < .094832,
\]

since the rational function decreases for \( 5 < m \). The function \( \varphi_{mn}(1) \) is 1, but for \( r > 1 \), then \( \varphi_{mn}(r) \) in (4.11) are bounded by rational functions which increase for
$m \geq 5$, and which we replace by their limits as $m \to \infty$.

$$\varphi_{mn}(2) = (y_{11}^{(2)} - y_{12}^{(2)})/y_{11}^{(1)} = 3(m - 2)(m - 6)/(13m^2 - 8m + 27)$$

(4.14a) $$< 3/13 = .23077,$$

$$\varphi_{mn}(3) = (y_{11}^{(3)} - y_{12}^{(3)} + 2y_{13}^{(3)}/3 + y_{22}^{(3)}/2)/y_{11}^{(1)}$$

(4.14b) $$< 17(m^2 - 6m + 32)(m - 2)(m - 2.4)/(120)(13m^2/8 - m + 35/8)^2$$

$$< (17/120)(8/13)^2 = .05365.$$

Similar calculations yield

(4.14c) $$\varphi_{mn}(4) < (1/32)(8/13)^3 = .00728.$$

Since the coefficients of $(8/13)^{-1}$ in $\varphi_{mn}(r)$ decrease as $r$ increases, the remaining sum of $\varphi_{mn}(r)$ is $< 2.6\varphi_{mn}(4)$. Hence, (4.11) implies

$$\text{tr } M < .8734 + .095(1.23077 + .05365 + 3.6(.00728))$$

(4.15) $$< .8734 + .095(1.3107) < .998 < 1.$$ 

This proves Theorem 4.1. We check directly for $m = 2, 3, 4$ that

$$\text{tr } M_2(6) = 97/105, \quad \text{tr } M_3(14) = .95 + 1/7280,$$

(4.16) $$\text{tr } M_4(25) = .87755 + .09359 + .0073 + .000005 < .9719.$$

This proves the parabolic bound $n > f(m) = (13m^2 - 5)/8$ to be sufficient for $\text{tr } M < 1$. Although some bound between this and $n > g(m)$ in (4.2) might also suffice for all $n$, the tight inequality (4.15) indicates that it would be difficult to prove.

5. The Bernoulli Continued Fraction. The entries $S_{i+j-2}(n)/n$ of the matrix $W_m(n)/n$ have as constant terms the Bernoulli numbers $B_{i+j-2}$ given in (1.2). The limit as $n \to 0$ of the leading principal minor of $W_m(n)/n$ is the determinant $b_{m-1}^*$ of order $m - 1$ expressible as

$$b_{m-1}^* = \det(B_{i+j}) = \lim_{n \to 0} (nb_{11})(n^{-m}w_m(n)).$$

Recalling $b_{11}$ from (4.3), $w_m(n)$ from (1.3), $v_m$ from (2.3) and $h_m$ from (2.4), we have

$$\lim_{n \to 0} nb_{11} = \binom{m}{m}/\binom{-1}{m-1} = (-1)^{m-1}m,$$

(5.2) $$\lim_{n \to 0} n^{-m}w_m(n) = h_mv_m^2(-1)^{m(m-1)/2},$$

(5.3)
\( b_{m-1}^* = (-1)^{(m-1)(m-2)/2}m!u_{m}/u_{2m}, \)

\( b_{m}^*/b_{m-1}^* = (-1)^{m-1}(m-1)!(m+1)!/(2m)!(2m+1)!. \)

Since \( B_{i+j} = 0 \) for odd \( i + j \), we can rearrange rows and columns of the matrix \((B_{i+j})\) so the odd numbered ones precede the even numbered ones, and thus factor \( b_{m-1}^* \) as the product \( d_{m-1}d_{m-2} \) of two determinants, where

\[
d_{2k-1} = \begin{vmatrix}
B_2 & B_4 & \cdots & B_{2k} \\
B_4 & B_6 & \cdots & B_{2k+2} \\
\vdots & \vdots & \ddots & \vdots \\
B_{2k} & B_{2k+2} & \cdots & B_{4k-2}
\end{vmatrix},
\]

(5.6)

\[
d_{2k} = \begin{vmatrix}
B_4 & B_6 & \cdots & B_{2k+2} \\
B_6 & B_8 & \cdots & B_{2k+4} \\
\vdots & \vdots & \ddots & \vdots \\
B_{2k+2} & B_{2k+4} & \cdots & B_{4k}
\end{vmatrix},
\]

(5.7)

\[ d_{m}/d_{m-2} = b_{m}^*/b_{m-1}^* \]

\[ -d_{m-3}d_{m}/d_{m-1}d_{m-2} = (m-1)m^2(m+1)/(2m-1)(2m)(2m+1) \]

(5.8)

\[ = (1/4)((m-1)m/(2m-1))(m(m+1)/(2m+1)). \]

**Theorem 5.1.** The divergent asymptotic alternating series

(5.9) \[ B(x) = \sum_{k=1}^{\infty} B_{2k}(2x)^{2k} = 4x^2/6 - 16x^4/30 + 64x^6/42 \cdots \]

has the convergent continued fraction expansion (1.6).

**Proof.** By the general theory of continued fractions [2], [9], if a formal power series (5.9) with arbitrary coefficients \( B_{2k} \) is expanded into continued fractions of the form

(5.10) \[ \frac{a_1(2x)^2}{1 + | \frac{a_2(2x)^2}{1 + \frac{x^2/c_0}{c_1 + \frac{x^2}{c_2 + \frac{x^2}{c_3 + \cdots}}}} \}} \]

and if the \( d_k \)'s are defined by (5.6), then
For the Bernoulli series Eqs. (5.5) and (5.11) imply

\begin{equation}
(5.12) \quad c_m = (m(m + 1)/(2m + 1))^{-1} = 1/m + 1/(m + 1), \quad m \geq 1,
\end{equation}

while the condition \(1/c_0c_1 = 4B_1 = 2/3\) implies \(c_0 = 1\). Since \(\Sigma c_m\) is divergent, the continued fraction (1.6) converges, and Theorem 5.1 is proved.

We can apply this continued fraction to approximate \(\pi^2\). It would require about a billion terms of the series \(\Sigma_{i=1}^\infty (1/k^2)\) to approximate \(\pi^2/6\) to nine decimals. But the Euler-Maclaurin summation formula gives the remainder after 5 terms by the expression

\begin{equation}
(5.13) \quad \int_6^\infty x^{-2} dx + 1/2 \cdot 6^2 + \sum_{k=1}^\infty B_{2k}(1/6)^{2k+1}.
\end{equation}

This alternating series diverges, with minimum remainder of about \(10^{-15}\) after the 19th term. Using the convergent continued fraction instead, we have

\begin{equation}
(5.14) \quad \pi^2 = 6(1 + 1/4 + 1/9 + 1/16 + 1/25 + 1/6 + 1/72) + B(1/12)
\end{equation}

\begin{equation}
(5.15) \quad \pi^2 = 9.865 + \frac{12^{-2}}{|1 + 1/2|} + \frac{12^{-2}}{|1/2 + 1/3|} + \frac{12^{-2}}{|1/3 + 1/4|} + \frac{1}{|1 + 1/2 + 1/6 + 1/2 + 1/6 + 1/4 + 1/3 + 1/3 + 2.4 + 2.4 + 2 + 2 + 2 + 2 + 2|},
\end{equation}

where the sixth convergent with \(r = 12/7\) has an error about \(10^{-12}\), and the tenth convergent (which changes this \(r\) to 1.9976) has an error less than \(10^{-15}\), giving \(\pi^2 = 9.869604401089359\).

The function \(s^{-1}B(s^{-1})\) is the Laplace transform of \(x \coth x - 1\). Continued fractions for the Laplace transforms of \(\tanh x\), \(\sech x\) and \(x \csch x\) can also be obtained by similar methods, but have already been derived by Stieltjes [8] and others, and are listed by Wall [9, p. 369]. The author has not found the continued fraction (1.6) in the literature, nor the determinantal formula (5.4) which evaluates the first principal \(m \times m\) minor \(b^*_m = |B_{i+j}|, i, j = 1, \ldots, (\text{omitting } B_0 \text{ and } B_1)\) of the determinant \(|B_{i+j-2}|\) of order \(m + 1\) called \(\Delta_m(B)\) by Al-Salam and Carlitz [1, p. 93, (3.1)] which in the notation of (2.3) becomes

\begin{equation}
(5.16) \quad \Delta_m(B) = (-1)^m(m + 1/2)(m!!)^6/(2m + 1)!!.
\end{equation}

Comparing (5.16) with (5.4) for order \(m\), we have

\begin{equation}
(5.17) \quad |B_{i+j}|_m = (-1)^m(m + 1)|B_{i+j-2}|_{m+1}.
\end{equation}