The Hankel Power Sum Matrix Inverse and the Bernoulli Continued Fraction

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Abstract. The \( m \times m \) Hankel power sum matrix \( W = VV^T \) (where \( V \) is the \( m \times n \) Vandermonde matrix) has \((i, j)\)-entry \( S_{i+j-2}(n) \), where \( S_p(n) = \sum_{k=1}^{n} k^p \). In solving a statistical problem on curve fitting it was required to determine \( f(m) \) so that for \( n > f(m) \) all eigenvalues of \( W^{-1} \) would be less than 1. It is proved, after calculating \( W^{-1} \) by first factoring \( W \) into easily invertible factors, that \( f(m) = (13m^2 - 5)/8 \) suffices. As by-products of the proof, close approximations are given for the Hilbert determinant, and a convergent continued fraction with \( m \)th partial denominator \( m^{-1} + (m + 1)^{-1} \) is found for the divergent Bernoulli number series \( \sum B_{2k}(2x)^{2k} \).

1. Introduction. Defined as the product \( W = VV^T \) of the \( m \times n \) Vandermonde matrix \( V = (j^{i-1}) \) with its transpose \( V^T \), the \( m \times m \) Hankel power sum matrix \( W = W_m \) has \((i, j)\)-entry \( S_{i+j-2}(n) \), where

\[
S_p = S_p(n) = \sum_{k=1}^{n} k^p = \frac{n^{p+1}}{p+1} + \frac{n^p}{2} + \sum_{k=1}^{\lfloor p/2 \rfloor} \frac{B_{2k}}{2k} \left( \frac{p}{2k-1} \right),
\]

and where \( B_{2k} \) are the Bernoulli numbers [3], [6]:

\[
B_2 = 1/6, \quad B_4 = -1/30, \quad B_6 = 1/42, \quad B_8 = -1/30, \quad B_{10} = 5/66, \ldots.
\]

In solving a statistical problem involving the fitting of polynomial curves of degree \( m \) to \( n > m \) points, for increasing \( m \) and \( n \), it was required [4] to find a function \( f(m) \), such that, whenever \( n > f(m) \), the eigenvalues \( \mu_k \) of \( M = W^{-1} \) would all be less than 1. We evaluate \( w_m = \det W_m \) in Section 2 as

\[
w_m = \det W_m = h_m \prod_{i,j=1}^{m} (n + i - j),
\]

where \( h_m \) is the determinant of the Hilbert matrix \( H_m \), and we obtain close estimates for \( h_m \). In Section 3 we factor \( W_m \) as a product of easily invertible matrices of which only diagonal matrices involve \( n \), and we also explicitly invert \( W_m \) and \( H_m \). In Section 4 we estimate the trace of \( M = W^{-1} \) and find that the function

\[
f(m) = (13m^2 - 5)/8
\]
suffices for powers of \( M \) to converge when \( n > f(m) \).

As a by-product of this investigation, we find in Section 5 that the divergent

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asymptotic series $B(x)$ with Bernoulli number coefficients

\[(1.5) \quad B(x) = \sum_{k=1}^{\infty} B_{2k}(2x)^{2k},\]

related to the Laplace transform of $x \coth x - 1$, has the convergent continued fraction expansion

\[(1.6) \quad B(x) = \frac{x^2}{1 + 1/2 + \frac{x^2}{1/2 + 1/3 + \frac{x^2}{1/3 + 1/4} + \cdots}}.\]

In fact, $B(1/12) = \pi^2 - 9.865$ is given with error $< 2 \times 10^{-12}$ by the sixth convergent of this continued fraction.

2. The Determinants. Since $S_0(n) = n$ and $n(n+1)/2$ divides $S_k(n)$ for $k > 0$, it follows directly that $w_m = \det W_m$ has the algebraic factor $n^m(n+1)^{m-1}$. For $r < m$, $(n-r)^m$ is also an algebraic factor of $w_m$, since the matrix $W_m(n)$ has rank $r$ and nullity $m-r$, when $n = r$ for $r = 1, 2, \ldots, m-1$. Since the polynomials $S_p(n)$ are generated by the function

\[(2.1) \quad G(x, n) = (e^{xn} - 1)/(1 - e^{-x}) = \sum_{p=0}^{\infty} S_p(n)x^p/p!,\]

we see from the identity

\[G(x, n) + G(-x, -n-1) + 1 = 0\]

that

\[(2.2) \quad S_p(n) + (-1)^p S_p(-n-1) + \delta_{p,0} = 0.\]

Hence, $w_m$ has the algebraic factor $(n+1+r)^{m-1-r}$. We have found $m^2$ linear functions of $n$ as factors of the polynomial $w_m(n)$ which is of degree $m^2$ in $n$. The remaining factor is the determinant of the leading coefficients $1/(i+j-1)$, namely the determinant $h_m$ of the ill-conditioned Hilbert matrix $H_m$ of order $m$ [5], [7]. This proves Eq. (1.3).

If we take $m = n$ in (1.3), the Vandermonde matrix $V$ in $W = VV^T$ is square. Its determinant $v_m$ is

\[(2.3) \quad v_m = \det V_m = 1! \ 2! \ 3! \ \cdots \ (m-1)! \equiv (m-1)!.\]

Hence by (1.3) and (2.3)

\[(2.4) \quad h_m = \det V_m = \left(\prod_{i,j=1}^{m} (m+i-j) = v_m^a/v_{2m}^a.\right)\]

The ratio of successive $h_m$'s is

\[(2.5) \quad h_m/h_{m+1} = (2m+1)!(2m)!/(m!)^2 = (2m+1)\left(\frac{2m}{m}\right)^2.\]

The following theorem gives a close approximation for $h_m$.

**Theorem 2.1.** The determinant $h_m$ of the Hilbert matrix $H_m$ of order $m$ is
given to 10 significant figures for \( m > 4 \) by

\[
h_m = 4^{-m(m-1)}(\pi/2)^{m-1}m^{-1/4}\exp R_m,
\]

where the remainder function \( R_m \) is defined by

\[
R_m = \int_0^\infty (e^{-2t} - e^{-2mt}) \tanh^2 (t/2)(4t)^{-1} dt
\]

and is approximated to 9 decimals for \( m \geq 5 \) by

\[
R_m = 0.013081539 - 2^{-6}m^{-2} + 2^{-8}m^{-4} - 2^{-8.5}m^{-6} + 2^{-8}m^{-8} - 2^{-7}m^{-10}.
\]

Proof. The ratio \( h_m/h_{m+1} \) in (2.5) is related to the Wallis approximation \( \pi_m \) for \( \pi \) by

\[
\frac{\pi_m}{2} = \frac{2}{1} \frac{2}{3} \frac{4}{3} \frac{4}{5} \cdots \frac{2m}{2m-1} \frac{2m}{2m+1} = 2^{4m}h_{m+1}/h_m.
\]

If we express \( \ln n \) in the form

\[
\ln n = \int_1^n \frac{ds}{s} = \int_0^\infty \int_1^n e^{-st} ds dt = \int_0^\infty (e^{-t} - e^{-nt}) t^{-1} dt,
\]

then \( \ln(\pi_m/2) \) and its limit \( \ln(\pi/2) \) are expressible as

\[
\ln(\pi_m/2) = \int_0^\infty (e^{-t} - 2e^{-2t} + 2e^{-3t} - 2e^{-2mt} + e^{-(2m+1)t}) t^{-1} dt
\]

\[
= \int_0^\infty (e^{-t} - e^{-(2m+1)t})(1-e^{-t})(1+e^{-t})^{-1} t^{-1} dt,
\]

\[
\ln(\pi/2) = \int_0^\infty e^{-t} \tanh (t/2)t^{-1} dt,
\]

\[
\ln(\pi/\pi_k) = \int_0^\infty e^{-(2k+1)t} \tanh (t/2)t^{-1} dt,
\]

\[
(\pi/2)^{m-1} \prod_{k=1}^{m-1} (\pi_k/\pi) = 2^{2m(m-1)}h_m/h_1.
\]

Summing in (2.13) from \( k = 1 \) to \( m - 1 \) yields

\[
\ln[(\pi/2)^{m-1}2^{-2m(m-1)}h_m] = \int_0^\infty (e^{-2t} - e^{-2mt})(e^{t/2} + e^{-t/2})^{-2} t^{-1} dt
\]

\[
= (1/4)\ln(2m/2) - R_m
\]

by (2.10), where \( R_m \) is defined by (2.7). Equation (2.6) follows from (2.15). To obtain (2.8) we evaluate \( R_4 = 0.012119610988 \) from (2.6) setting \( h_4 = 1/6048000 \) in (2.6). Then we compute \( R_m - R_m \) from (2.7) by replacing \( \tanh(t/2) \) by the first five terms of its series, and set \( m = 4 \) to get \( R_m \) in (2.8). We check the tenth decimal by working from \( h_5 \) instead. This gives exp \( R_m \) and \( h_m \) accurate to 10 significant figures.
For \( m = 20 \) we find \( R_{20} = 0.0130425009 \) and

\[
(2.16) \quad h_{20} = 4.206178954 \times 10^{-226}.
\]

The matrices \( H_m \) and \( W_m(n) \) are ill conditioned. In fact, \( W_3(3) \) has the eigenvalues \( \lambda_1 = 113.4132, \lambda_2 = 1.564253, \lambda_3 = 0.02254695 \) and the conditioning ratio \( \lambda_1/\lambda_3 = 5030 \). So the usual computer methods for inverting \( W_m(n) \) are unreliable [5], [7].

3. Inversion by Factoring. To invert the ill-conditioned \( m \times m \) matrix \( W = \begin{pmatrix} S_{i+j-2}(n) \end{pmatrix} \) with \((i, j)\)-entry \( S_{i+j-2}(n) \), we first factor it into easily invertible factors, restricting the variable \( n \) to diagonal matrix factors \( EP = (\text{diag } e_i) \) and \( Q = (\text{diag } q_j) \), where

\[
(3.1) \quad e_i = (-1)^{i-1}, \quad p_i = \binom{n + i - 1}{n - m}, \quad q_j = \binom{n - j}{n - m}.
\]

We denote by \( T = (t_{ij}) \) the lower Pascal triangle matrix with

\[
(3.2) \quad t_{ij} = \binom{i - 1}{j - 1} = \binom{i - 1}{i - j} = (-1)^{i+j+1} \binom{-j}{i - j}.
\]

We note that \( ETE \) has entries \( (-1)^{i+j} \), so

\[
(3.3) \quad (TETE)_{ij} = \sum_{k=j}^{i} \binom{i-1}{i-k} \binom{-j}{k-j} = \binom{i-j-1}{i-j} = \delta_{ij}
\]

and \( T^{-1} = ETE \). Next, we define an \( m \times m \) lower triangular row stochastic matrix \( A = (a_{ij}) \) that converts the integral powers in \( V \) into binomial coefficients by the formula

\[
(3.4) \quad (AV)_{ik} = \sum_{r=1}^{i} a_{ir} k^{-1} = \binom{k + i - 2}{i - 1}, \quad k = 1, 2, \ldots, n.
\]

The \( a_{ir} \)'s are related to Stirling numbers of the first kind, and

\[
(3.5) \quad A = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & \cdots \\
0 & 1 & 0 & 0 & 0 & \cdots \\
0 & 1/2! & 1/2! & 0 & 0 & \cdots \\
0 & 2/3! & 3/3! & 1/3! & 0 & \cdots \\
0 & 6/4! & 11/4! & 6/4! & 1/4! & \cdots
\end{bmatrix} = (a_{ij}).
\]

From (3.5) and (3.4) we obtain

\[
(3.6) \quad (AV)_{ik} = \binom{i + k - 2}{i - 1} = \sum_{j=1}^{i} \binom{i-1}{i-j} \binom{k-1}{j-1} = \sum_{j=1}^{k} t_{ij} \binom{k-1}{j-1}.
\]

To factor \( W \) we now write
\[ (AV(T^{-1}AV)^T)_{ij} = \sum_{k=j}^{n} \binom{k+i-2}{i-1} \binom{k-1}{j-1} \]

\[ = \binom{i+j-2}{i-1} \sum_{k=j}^{n} \binom{k+i-2}{j+i-2}. \]

Summing over \( k \) yields

\[ (AW(T^{-1}A)^T)_{ij} = \binom{i+j-2}{i-1} \binom{n+i-1}{j+i-1} \]

\[ = p_i \binom{i+j-2}{i-1} \binom{m+i-1}{j+i-1} / q_j. \]

**Theorem 3.1.** The inverse matrix \( M = W^{-1} \) has the factorization

\[ M = W^{-1} = A^T E B A, \quad B = T^T Q T T^T T P^{-1}, \]

where \( E, P, Q, T, A \) are defined in (3.1), (3.2) and (3.5).

**Proof.** Entries of \( TT^T \) and \( TT^T T \) are

\[ (TT^T)_{ik} = \sum_{r=1}^{i} \binom{i-1}{i-r} \binom{k-1}{r-1} = \binom{i+k-2}{i-1}, \]

\[ (TT^T T)_{ij} = \sum_{k=j}^{m} \binom{i+k-2}{k-1} \binom{k-1}{j-1} = \binom{i+j-2}{i-1} \sum_{k=j}^{m} \binom{k+i-2}{j+i-2}. \]

Summing over \( k \) yields

\[ (TT^T T)_{ij} = \binom{i+j-2}{i-1} \binom{m+i-1}{j+i-1}. \]

Combining (3.8) and (3.12), we have

\[ (AWA^T)^{-1} = PTT^T Q T^{-1} T. \]

Since the diagonal sign matrix \( E \) commutes with \( P \) and \( Q \) but transforms \( T \) and \( T^T \) into their inverses,

\[ (AWA^T)^{-1} = E T^T Q T T^T T P^{-1} E = E B E \]

and (3.9) is proved.

Equation (3.12) provides a simple method for inverting the Hilbert matrix \( H \).

**Theorem 3.2.** The inverse of the \( m \times m \) Hilbert matrix \( H = (h_{ij}) \) with \( (i, j) \)-entry \( h_{ij} = 1/(i + j - 1) \) is given by

\[ (H^{-1})_{ij} = d_i d'_j h_{ij} d_j d'_j, \]

\[ d'_i = \binom{-m-1}{i-1}, \quad d_j = \binom{m-1}{j-1}. \]
Proof. Factoring (3.12) yields

\[(3.16) \quad (ETT^T)^m_{ij} = (-1)^{i-1} \binom{m + i - 1}{i - 1} \frac{m}{i + j - 1} \binom{m - 1}{j - 1} = d_i^j h_{ij} d_j.
\]

Since \(ETT^T\) is involutory, \(D'HD = (D'HD)^{-1}\). Also,

\[(3.17) \quad h_m = \det H_m = \pm 1 \prod_{i=1}^m d_i d_i^j.
\]

Although the matrix \(B\) in (3.9) is symmetric, its symmetry is not obvious from formula (3.9).

Theorem 3.3. The symmetric matrix \(B = (b_{ij})\) in (3.9) has entries expressible in terms of descending factorials \((x)_r = x(x - 1) \cdots (x - r + 1)\) as follows:

\[(3.18) \quad b_{ij} = \sum_{r \geq i + j - 1} \frac{(m + i - 1)_r(m + j - 1)_r}{(n - m + r)_r r!} \binom{r - 1}{i - 1, j - 1},
\]

where \(\binom{r - 1}{i - 1, j - 1}\) is the trinomial coefficient \(\binom{r - 1}{i - 1, j - 1}\).

Proof. To transform \(B\) in (3.9) we evaluate

\[(3.19) \quad (T^T Q TT^T)_{is} = \sum_{r=i}^m \binom{r - 1}{i - 1} \binom{n - r}{n - m} \binom{r + s - 2}{r - 1}
\]

\[= \binom{i + s - 2}{i - 1} \sum_{r=i}^m \binom{r + s - 2}{i + s - 2} \binom{n - r}{n - m}
\]

\[= \binom{i + s - 2}{i - 1} \binom{n + s - 1}{m - i},
\]

\[(BP)_{ij} = \sum_{s=j}^m \binom{i + s - 2}{i - 1} \binom{n + s - 1}{m - i} \binom{s - 1}{j - 1}
\]

\[= \binom{i + j - 2}{i - 1} \sum_{s=j}^m \binom{i + s - 2}{i - s - j} \sum_{r \geq i + j - 1} \binom{s - j}{r - i - j + 1} \binom{n + j - 1}{m + j - r}
\]

\[= \sum_{r \geq i + j - 1} \sum_{s=j}^m \binom{i + s - 2}{i - 1} \binom{r - 1}{n - m + r} \binom{n + j - 1}{n - m + r}.
\]

Summing over \(s\) and dividing by \(p_j\), we have

\[(3.20) \quad b_{ij} = \sum_{r \geq i + j - 1} \binom{m + i - 1}{r} \binom{r - 1}{i - 1, j - 1} \binom{m + j - 1}{n - m + r}.
\]

Writing \((x)_r = r!(x/r)\), Eq. (3.21) becomes (3.18).
To conserve space in displaying the symmetric matrices $M_m(n) = W_m^{-1}(n)$ we show the upper half of $M_3$ and the lower half of $M_4$.

(3.22)

$$
\begin{bmatrix}
16n^2 + 24n^2 + 56n + 24 \\
-120(n^2 + n) - 100
\end{bmatrix}
$$

(3.3)

4. Estimation of $\text{tr} M$. Since $W$ and $M$ are positive definite for $n > m$, all eigenvalues $\mu_k$ of $M$ will satisfy $\mu_k < 1$ if $\text{tr} M < 1$, $m > 1$. For $A$ in (3.5) and $e_1 = (-1)^{t-1}$, the first and last diagonal entries of $M = M_m$ are $b_{11}$ and $b_{mm} / ((m-1)!)^2$. Numerical computation shows that the maximum $n$ for which $\det(W_m(n) - I) = 0$ are given for $m = 1, 2, 3, 4$ by

(4.1) $(m, n) = (1, 1), (2, 5.82090), (3, 13.3776), (4, 24.24453)$.

The parabola through the first three points is

(4.2) $n = g(m) = 1.5679m^2 - .0828m - .4851$,

and we find $g(4) = 24.270 > 24.24453$. A slightly higher value than (4.2) will be required for $\text{tr} M < 1$. We first estimate the dominant diagonal entry $b_{11}$ of $M$.

$$
\binom{n}{m} b_{11} = \sum_{r=1}^{m} \binom{m}{r} \left( \frac{n}{m} \right)^{n-r} \left( \frac{m}{m-r} \right)^{r} = \sum_{r=1}^{m} \binom{m}{r} \binom{n}{m-r}
$$

(4.3)

$$
1 + b_{11} = \left( \frac{n+m}{m} \right)^n \left( \frac{m}{m} \right)^{n-m} = \prod_{k=1}^{m} \frac{2n + 1 + (2k - 1)}{2n + 1 - (2k - 1)},
$$

(4.4)

(4.5a) $\ln(1 + b_{11}) = \sum_{k=1}^{m} \ln \left( \frac{1 + (2k - 1)/(2n + 1)}{1 - (2k - 1)/(2n + 1)} \right) = \sum_{r=1}^{m} \frac{\theta(m, r)}{(2r-1)(2n+1)^{2r-1}}$.

where

(4.5b) $\theta(m, r) = \sum_{k=1}^{m} 2(2k-1)x^{2r-1} < \int_{0}^{2m} x^{2r-1} dx = (2m)^{2r}/2r$.

We now assume the inequalities $n > f(m)$ in (1.4).

**Theorem 4.1.** The matrix $M = W_m^{-1}(n)$ has trace $< 1$ if

(4.6) $n > 1.625m^2 - .625$ and $m > 5$. 

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Proof. If (4.6) is satisfied for \( m = 5 \), then \( m \geq 40 \), and

\[
(4.7a) \quad b_{11} \leq (45)_5/(40)_5 - 1 = 62639/73112 = 0.856754.
\]

If (4.6) is satisfied for \( m \geq 6 \), then \((2n + 1)/2m^2 > 1.6215\) and

\[
\ln(1 + b_{11}) < \frac{2m^2}{2n + 1} \sum_{r=1}^{\infty} \left( \frac{2m}{2n + 1} \right)^{2r-2} / r(2r - 1)
\]

\[
(4.8) \quad < \frac{1}{1.6215} \sum_{r=1}^{\infty} \left( \frac{1}{9.729} \right)^{2r-2} / r(2r - 1) < 0.62777,
\]

\[(4.7b) \quad b_{11} < 0.8734 \quad \text{for} \quad m \geq 6, \quad n > (13m^2 - 5)/8.\]

The rest of \( \text{tr} M \) is given by

\[
(4.9) \quad \text{tr} M - b_{11} = \sum_{k=2}^{m} \sum_{i,j=k}^{m} a_{ik}(-1)^i b_{ij}(-1)^j a_{jk}.
\]

We replace \( i, j, r \) by \( i + 1, j + 1, r + 2 \) and write

\[
(4.10) \quad b_{i+1,j+1} = \sum_{k=1}^{m-1} y_{ij}^{(r)}, \quad y_{ij}^{(r)} = \frac{(m + i)_{r+2}(m + j)_{r+2}}{(n - m + r + 2)_{r+2}(r + 2)!} \binom{r + 1}{i,j} = y_{ij}^{(r)}.
\]

Then

\[
(4.11) \quad \text{tr} M - b_{11} = y_{11}^{(1)} \sum_{r=1}^{2m-1} \varphi_{mn}(r), \quad \varphi_{mn}(r) = \sum_{i+j=2}^{r+1} c_{ij} y_{ij}^{(r)} / y_{11}^{(1)},
\]

where the entries of the \((m - 1) \times (m - 1)\) matrix \( C = (c_{ij}) \) are

\[
(4.12a) \quad c_{ij} = (-1)^{i+j} \sum_{k=1}^{m-1} a_{i+1,k} a_{j+1,k} = c_{ji},
\]

\[
(4.12b) \quad C = \frac{1}{720} \begin{bmatrix}
720 & -360 & 240 & -180 & 144 & \cdots \\
-360 & 360 & -300 & 255 & -222 & \cdots \\
240 & -300 & 280 & -255 & 233 & \cdots \\
-180 & 255 & -255 & 242.5 & -228.5 & \cdots \\
144 & -222 & 233 & -228.5 & 220.1 & \cdots \\
\end{bmatrix}
\]

The dominant term \( y_{11}^{(1)} \) satisfies

\[
(4.13) \quad y_{11}^{(1)} < \frac{(m + 1)_{3}(m + 1)_3/3}{(13m^2/8 - m + 19/8)_3} \leq \frac{6363/3}{(38)_3} = \frac{200}{2109} < 0.094832,
\]

since the rational function decreases for \( 5 \leq m \). The function \( \varphi_{mn}(1) \) is 1, but for \( r > 1 \), then \( \varphi_{mn}(r) \) in (4.11) are bounded by rational functions which increase for
\( m \geq 5 \), and which we replace by their limits as \( m \to \infty \).

\[
\varphi_{mn}(2) = (y_{11}^{(2)} - y_{12}^{(2)})/y_{11}^{(1)} = 3(m - 2)(m - 6)/(13m^2 - 8m + 27)
\]

\( (4.14a) \)

\[
< 3/13 = .23077,
\]

\[
\varphi_{mn}(3) = (y_{11}^{(3)} - y_{12}^{(3)} + 2y_{13}^{(3)}/3 + y_{22}^{(3)}/2)/y_{11}^{(1)}
\]

\( (4.14b) \)

\[
< 17(m^2 - 6m + 32)(m - 2)(m - 2.4)/(120)(13m^2/8 - m + 35/8)^2
\]

\[
< (17/120)(8/13)^2 = .05365.
\]

Similar calculations yield

\[
\varphi_{mn}(4) < (1/32)(8/13)^3 = .00728.
\]

Since the coefficients of \((8/13)^{-1}\) in \( \varphi_{mn}(r) \) decrease as \( r \) increases, the remaining sum of \( \varphi_{mn}(r) \) is \( < 2.6 \varphi_{mn}(4) \). Hence, (4.11) implies

\[
\text{tr} \, M < .8734 + .095(1.23077 + .05365 + 3.6(.00728))
\]

\( (4.15) \)

\[
< .8734 + .095(1.3107) < .998 < 1.
\]

This proves Theorem 4.1. We check directly for \( m = 2, 3, 4 \) that

\[
\text{tr} \, M_2(6) = 97/105, \quad \text{tr} \, M_3(14) = .95 + 1/7280,
\]

\( (4.16) \)

\[
\text{tr} \, M_4(25) = .87755 + .09359 + .0073 + .0000005 < .9719.
\]

This proves the parabolic bound \( n > f(m) = (13m^2 - 5)/8 \) to be sufficient for \( \text{tr} \, M < 1 \). Although some bound between this and \( n > g(m) \) in (4.2) might also suffice for all \( n \), the tight inequality (4.15) indicates that it would be difficult to prove.

5. The Bernoulli Continued Fraction. The entries \( S_{i+j-2}(n)/n \) of the matrix \( W_m(n)/n \) have as constant terms the Bernoulli numbers \( B_{i+j-2} \) given in (1.2). The limit as \( n \to 0 \) of the leading principal minor of \( W_m(n)/n \) is the determinant \( b_{m-1}^* \) of order \( m - 1 \) expressible as

\[
b_{m-1}^* = \det(B_{i+j}) = \lim_{n \to 0} (nb_{11})(n^{-m}w_m(n)).
\]

Recalling \( b_{11} \) from (4.3), \( w_m(n) \) from (1.3), \( v_m \) from (2.3) and \( h_m \) from (2.4), we have

\[
\lim_{n \to 0} nb_{11} = \binom{m}{m} / \binom{-1}{m-1} = (-1)^{m-1}m,
\]

\( (5.2) \)

\[
\lim_{n \to 0} n^{-m}w_m(n) = h_m v_m^2 (-1)^m (m-1)/2,
\]

\( (5.3) \)
\( b_{m-1}^* = (-1)^{(m-1)(m-2)}/2mv_m^6/v_{2m}, \)

\( b\, b_{m-1}^* = (-1)^{m-1}(m-1)!(m+1)/(2m)!(2m+1)! . \)

Since \( B_{i+j} = 0 \) for odd \( i + j \), we can rearrange rows and columns of the matrix \((B_{i+j})\) so the odd numbered ones precede the even numbered ones, and thus factor \( b_{m-1}^* \) as the product \( d_{m-1}d_{m-2} \) of two determinants, where

\[
d_{2k-1} = \begin{vmatrix} B_2 & B_4 & \cdots & B_{2k} \\ B_4 & B_6 & \cdots & B_{2k+2} \\ \vdots & \vdots & \ddots & \vdots \\ B_{2k} & B_{2k+2} & \cdots & B_{4k-2} \end{vmatrix},
\]

\[
d_{2k} = \begin{vmatrix} B_4 & B_6 & \cdots & B_{2k+2} \\ B_6 & B_8 & \cdots & B_{2k+4} \\ \vdots & \vdots & \ddots & \vdots \\ B_{2k+2} & B_{2k+4} & \cdots & B_{4k} \end{vmatrix},
\]

\[
d_{m}/d_{m-2} = b_{m-1}^*/b_{m-1}^* .
\]

\[
-d_{m-3}d_{m}/d_{m-1}d_{m-2} = (m-1)m^4/(2m-1)(2m+1)
\]

\[
= (1/4)((m-1)m/(2m-1))(m(m+1)/(2m+1)).
\]

**Theorem 5.1.** The divergent asymptotic alternating series

\[
B(x) = \sum_{k=1} B_{2k}(2x)^{2k} = 4x^2/6 - 16x^4/30 + 64x^6/42 \cdots
\]

has the convergent continued fraction expansion (1.6).

**Proof.** By the general theory of continued fractions [2], [9], if a formal power series (5.9) with arbitrary coefficients \( B_{2k} \) is expanded into continued fractions of the form

\[
\frac{a_1(2x)^2}{1 + \frac{a_2(2x)^2}{1 + \frac{x^2/c_0}{c_1 + \frac{x^2}{c_2 + \frac{x^2}{c_3 + \cdots}}}}}
\]

and if the \( d_k \)’s are defined by (5.6), then
For the Bernoulli series Eqs. (5.5) and (5.11) imply

\[ c_m = (m(m + 1)/(2m + 1))^{-1} = 1/m + 1/(m + 1), \quad m \geq 1, \]

while the condition \(1/c_0 c_1 = 4B_1 = 2/3\) implies \(c_0 = 1\). Since \(\Sigma c_m\) is divergent, the continued fraction (1.6) converges, and Theorem 5.1 is proved.

We can apply this continued fraction to approximate \(\pi^2\). It would require about a billion terms of the series \(\Sigma \alpha(1/k^2)\) to approximate \(\pi^2/6\) to nine decimals. But the Euler-Maclaurin summation formula gives the remainder after 5 terms by the expression

\[ \int_6^\infty x^{-2} dx + 1/2 \cdot 6^2 + \sum_{k=1}^\infty B_{2k}(1/6)^{2k+1}. \]

This alternating series diverges, with minimum remainder of about \(10^{-15}\) after the 19th term. Using the convergent continued fraction instead, we have

\[ \pi^2 = 6(1 + 1/4 + 1/9 + 1/16 + 1/25 + 1/6 + 1/72) + B(1/12) \]

\[ = 9.865 + \frac{12^{-2}}{1 + 1/2} + \frac{12^{-2}}{1/2 + 1/3} + \frac{12^{-2}}{1/3 + 1/4}, \]

\[ = 9.865 + \frac{1/12}{12 + 6} + \frac{1}{6 + 4} + \frac{1}{4 + 3} + \frac{1}{3 + 2.4} + \frac{1}{2.4 + 2} + \frac{1}{2 + \tau}, \]

where the sixth convergent with \(r = 12/7\) has an error about \(10^{-12}\), and the tenth convergent (which changes this \(r\) to 1.9976) has an error less than \(10^{-15}\), giving \(\pi^2 = 9.869604401089359\).

The function \(s^{-1}B(s^{-1})\) is the Laplace transform of \(x \coth x - 1\).

Continued fractions for the Laplace transforms of \(\tanh x\), \(\sech x\) and \(x \csch x\) can also be obtained by similar methods, but have already been derived by Stieltjes [8] and others, and are listed by Wall [9, p. 369]. The author has not found the continued fraction (1.6) in the literature, nor the determinantal formula (5.4) which evaluates the first principal \(m \times m\) minor \(B_m = |B_{ij}|, i, j = 1, \ldots, m\), (omitting \(B_0\) and \(B_1\)) of the determinant \(|B_{i+j-2}|\) of order \(m + 1\) called \(\Delta_m(B)\) by Al-Salam and Carlitz [1, p. 93, (3.1)] which in the notation of (2.3) becomes

\[ \Delta_m(B) = (-1)^m(m+1)/2(m!!)^6/(2m+1)!!. \]

Comparing (5.16) with (5.4) for order \(m\), we have

\[ |B_{i+j}|m = (-1)^m(m+1)|B_{i+j-2}|m+1. \]