Two Conjectures of B. R. Santos Concerning Totitives

By H. G. Kopetzky and W. Schwarz

Abstract. Recently B. R. Santos conjectured that 12 is the largest integer n with the following property:

\[(*) \quad \begin{cases} 
\text{If } m \in [1, n] \text{ and } n \text{ are relatively prime, then } \\
\text{n + m is a prime number.}
\end{cases} \]

Using deep numerical estimates of Rosser and Schoenfeld for the number \(\pi(x)\) of primes less than \(x\), it is proved that the conjecture of Santos is true. The same result holds, if in addition it is assumed in \((*)\) that \(m\) is a prime.

The positive integers not greater than a given integer and coprime to it are called its totitives. It is well known that 30 is the largest integer with the property that all its totitives are prime.

B. R. Santos [4] proved that there exists a largest integer \(n\) with the property

\[(P_1) \quad 1 \leq m \leq n, \quad \gcd(m, n) = 1 \Rightarrow (n + m \text{ is prime}). \]

He conjectured that \(n = 12\) is the largest integer having the property \((P_1)\). Furthermore, he conjectures that there is a largest integer \(n\) with the property

\[(P_2) \quad 1 \leq m \leq n, \quad \gcd(m, n) = 1 \text{ and } m \text{ prime } \Rightarrow (n + m \text{ is prime}). \]

In this note we prove the following results.

(A) \(n = 12\) is the largest integer with the property \((P_1)\).
(B) \(n = 12\) is the largest integer with the property \((P_2)\).

Denote by \(\pi(x)\) the number of primes not greater than \(x\), and by \(\varphi(n)\) Euler's function.

(A) is true if

\[
\varphi(n) > \pi(2n) - \pi(n)
\]

holds for \(n > 12\). In order to prove (1) we use the following estimates due to Rosser and Schoenfeld [2, Theorems 1 and 15].

\[(RS1) \quad \pi(x) > f(x) := \frac{x}{\log x} \left(1 + \frac{1}{2 \log x}\right) \quad \text{for } x \geq 59, \]

\[(RS2) \quad \pi(x) < g(x) := \frac{x}{\log x} \left(1 + \frac{3}{2 \log x}\right) \quad \text{for } x > 1, \]

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and\ *

(RS 3) \[
\frac{n}{\varphi(n)} < e^{c \log \log n + \frac{5}{2 \log \log n}} \quad \text{for } n \geq 3,
\]

except when \( n = n_0 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 19 \cdot 23 \). For this special \( n \) the constant \( 5/2 \) in (RS 3) has to be replaced by \( 2.50637 \).

Define for \( x \geq 3 \) the function \( h(x) \) by

\[
h(x) = x \cdot \left( e^{c \log \log x + \frac{5}{2 \log \log x}} \right)^{-1}.
\]

A simple computation shows that the function \( F(x) = h(x) - g(2x) + f(x) \) has a zero between 139 and 140 and is increasing for \( x \geq 139 \). Using (RS 1), (RS 2) and (RS 3), we obtain

(2) \[
\varphi(n) > \pi(2n) - \pi(n) \quad \text{whenever } n \geq 139, \ n \neq n_0.
\]

For \( 12 < n \leq 139 \) and \( n = n_0 \) inequality (2) is verified numerically. Hence proposition (A) is true.

Denote by \( \omega(n) \) the number of different prime factors of \( n \). Proposition (B) is true, if the inequality

(3) \[
\pi(n) - \omega(n) > \pi(2n) - \pi(n)
\]

can be shown for \( n > n_1 \), and if (B) can be verified directly for \( 12 < n \leq n_1 \).**

Inequality (3) holds for all sufficiently large \( n \). This follows easily from the trivial estimate

(4) \[
\omega(n) \leq \frac{\log n}{\log 2}
\]

and Landau’s result (see [1])

\[
2 \pi(x) - \pi(2x) = 2 \log 2 \cdot \frac{x}{\log^2 x} + o\left( \frac{x}{\log^2 x} \right).
\]

Denote by \( \vartheta(x) \) the logarithm of the product of all primes not greater than \( x \). Connections between \( \vartheta(x) \) and \( \pi(x) \) are established by partial summation, for example

(5) \[
\pi(x) = \frac{\vartheta(x)}{\log x} + \int_2^x \frac{\vartheta(u)}{u \log^2 u} \, du.
\]

Rosser and Schoenfeld [3, Theorem 8] proved the inequality

(RS 4) \[
|\vartheta(x) - x| < 8.6853 \cdot \frac{x}{\log^2 x} \quad \text{for } x > 1.
\]

From (5) and (RS 4) we deduce

(6) \[
\left| \pi(x) - \frac{x}{\log x} - \frac{x}{\log^2 x} - \frac{2x}{\log^3 x} \right| < 9.5 \cdot \frac{x}{\log^3 x} \quad \text{for } x \geq 10^{10}
\]

\* \( C \) denotes Euler’s constant.

\** We prove (3) for \( n > 58 \).
by the same argument as in [2, Section 7]. *** Inequality (6) combined with (4) easily establishes the truth of (3) for \( n \geq 10^{11} \).

Now we use (5.1) and (5.2) from [3] to get

\[ |\delta(x) - x| < 0.001316x \quad \text{for} \quad x \geq 10^7; \]

hence

\[ |\delta(x) - x| < 0.85 \cdot \frac{x}{\log^2 x} \quad \text{for} \quad 10^7 \leq x \leq 10^{11}. \]

In the same way as before we obtain

(7) \[ \left| \pi(x) - \frac{x}{\log x} - \frac{x}{\log^2 x} - \frac{2x}{\log^3 x} \right| < 2 \cdot \frac{x}{\log^3 x} \quad \text{for} \quad 10^7 \leq x \leq 10^{11}. \]

This inequality enables us to deduce (3) for \( 10^7 \leq n \leq 10^{11} \).

In order to handle the interval \( n \leq 10^7 \) we use (4.1) and (4.2) from [2], giving

(RS 5) \[ \log x - \log x^{\frac{1}{2}} < \pi(x) < \log x \quad \text{for} \quad 11 \leq x \leq 10^8. \]

The function

\[ r(x) := 2 (\log x - \log x^{\frac{1}{2}}) - \log 2 \]

is increasing when \( x \geq 40 \), and \( r(x) - \log x/\log 2 \) is increasing for \( x \geq 2310 \). Since \( r(2310) > 35 > (1/\log 2) \cdot \log 2310 \), inequality (3) is true for \( 2310 \leq n \leq 10^8 \). Since \( \omega(n) < 4 \) when \( n < 2310 \) and \( r(500) > 4 \), inequality (3) is true for \( 500 \leq n \leq 2310 \). Numerical calculations give the truth of (3) for \( 58 < n < 500 \). In the interval \( 12 < n \leq 58 \) proposition (B) is verified directly. We remark that our calculations prove \( \pi(2x) < 2\pi(x) \) for \( x > 11 \). This result was announced by Rosser and Schoenfeld in the introduction to [3]. We further remark that the constants and the ranges of our estimates are not the best possible that can be obtained from the estimates of Rosser and Schoenfeld.

Finally, Santos asks whether there is a largest integer \( n \) with the property

(P3) \[ 1 \leq m \leq n, \quad \text{g.c.d.} (n, m) = 1 \text{ and } m \text{ composite } \Rightarrow (n + m \text{ is prime}). \]

Since the inequality

(8) \[ \varphi(n) - \pi(n) + \omega(n) > 0 \]

is true for \( n > 281, \dagger \) it is possible to show that \( n = 12 \) is the largest integer having property (P3).

***The values of the logarithmic integral \( \text{li}(x) \) needed for the proof can be calculated by using the power series expansion of the exponential integral \( E1(x) \) and the relation \( \text{li}(x) = E1(\log x). \)

\dagger This follows from (RS 1) and (RS 3).


3. J. B. ROSSER & L. SCHOENFELD, *Sharper Bounds for the Chebyshev Functions \( \theta(x) \) and \( \psi(x) \)*, University of Wisconsin MRC Technical Summary Report #1475, 1974.