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# On Computing Artin $L$-Functions in the Critical Strip

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Abstract. This paper gives a method for computing values of certain nonabelian Artin $L$-functions in the complex plane. These Artin $L$-functions are attached to irreducible characters of degree 2 of Galois groups of certain normal extensions $K$ of Q. These fields $K$ are the ones for which $G = \mathrm{Gal}(K/\mathrm{Q})$ has an abelian subgroup $A$ of index 2, whose fixed field $\mathrm{Q}(\sqrt{d})$ is complex, and such that there is a $\sigma \in G - A$ for which $\sigma a \sigma^{-1} = a^{-1}$ for all $a \in A$. The key property proved here is that these particular Artin $L$-functions are Hecke (abelian) $L$-functions attached to ring class characters of the imaginary quadratic field $\mathrm{Q}(\sqrt{d})$ and, therefore, can be expressed as linear combinations of Epstein zeta functions of positive definite binary quadratic forms. Such Epstein zeta functions have rapidly convergent expansions in terms of incomplete gamma functions.

In the special case $K = \mathrm{Q}(\sqrt{-3}, a^{1/3})$, where $a > 0$ is cube-free, the Artin $L$-function attached to the unique irreducible character of degree 2 of $\mathrm{Gal}(K/\mathrm{Q}) \cong S_3$ is the quotient of the Dedekind zeta function of the pure cubic field $L = \mathrm{Q}(a^{1/3})$ by the Riemann zeta function. For functions of this latter form, representations as linear combinations of Epstein zeta functions were worked out by Dedekind in 1879. For $a = 2$, 3, 6 and 12, such representations are used to show that all of the zeroes $\rho = \sigma + it$ of these $L$-functions with $0 < \sigma < 1$ and $|t| \leqslant 15$ are simple and lie on the critical line $\sigma = \frac{1}{2}$. These methods currently cannot be used to compute values of $L$-functions with $\mathrm{Im}(s)$ much larger than 15, but approaches to overcome these deficiencies are discussed in the final section.

**1. Introduction.** The locations of the complex zeroes of the Riemann zeta function and its generalizations play a central role in analytic number theory. For this reason extensive computations have been carried out locating zeroes of the Riemann zeta function [18], [22], and less extensive computations for Dirichlet $L$-functions [13], [28], [35], and certain Hecke $L$-functions with grossencharacters [12]. This paper discusses the computation of zeroes for certain members of another class of $L$-functions, the nonabelian Artin $L$-functions attached to normal extensions of the rational numbers Q. The emphasis is on checking whether or not all the zeroes $\rho = \sigma + it$ with $0 < \sigma < 1$ and $|t| < T$ (a chosen bound) lie on the critical line $\sigma = \frac{1}{2}$ (i.e., are consistent with the generalized Riemann hypothesis (GRH)), and on evaluating the multiplicities of such zeroes. This study was undertaken to test the (unlikely) possibility that any nonabelian Artin $L$-function attached to a nontrivial irreducible

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character of degree $n$ has multiple zeroes in the critical strip with multiplicities divisible by $n$. This possibility arose in an examination of A. I. Vinogradov's [34] suggested approach to proving Artin's conjecture that all such Artin $L$-functions are entire. For the Artin $L$-functions we computed, which are attached to irreducible characters of degree 2, all the zeroes in the regions investigated were on the critical line and were simple.

Artin $L$-functions and their properties are described in Section 2. The nonabelian Artin $L$-functions to which our approach applies are exactly those Artin $L$-functions which can be expressed as linear combinations of Epstein zeta functions of positive definite binary quadratic forms. In Section 3 we exhibit a large class of Galois extensions $K$ over $\mathbf{Q}$ all of whose nonabelian Artin $L$-functions attached to irreducible characters of degree 2 have this property. These fields $K$ are characterized in two related ways. First, they are exactly the fields over a complex quadratic field $\mathbf{Q}(\sqrt{d})$ which are normal subfields of some ring class field of $\mathbf{Q}(\sqrt{d})$. Second, Proposition 3.5 characterizes these fields as those normal extensions $K$ of $\mathbf{Q}$ such that $G = \mathrm{Gal}(K/\mathbf{Q})$ has an abelian subgroup $A$ of index 2 whose fixed field $\mathbf{Q}(\sqrt{d})$ is complex and for which there exists an element $\sigma \in G - A$ such that $\sigma a \sigma^{-1} = a^{-1}$ for all $a \in A$. (In particular, this includes all totally complex fields $K$ with $\mathrm{Gal}(K) \cong \mathbf{Z}_2 \oplus \mathbf{Z}_2$ or $\mathrm{Gal}(K) \cong S_3$.) This partially answers a question raised by Shanks [23], [27], who asked for a Galois-theoretic criterion for fields $K$ whose Dedekind zeta function can be written as a product of linear combinations of Epstein zeta functions. In fact, it is quite likely that these fields $K$ are the entire set of fields having this property. In Section 3 we also show that each nonabelian Artin $L$-function associated to such a field $K$ is a Hecke (abelian) $L$-function associated to a ring-class character of some imaginary quadratic field. In particular, Artin's conjecture holds for these Artin $L$-functions, i.e., they are entire functions.

The usefulness of expressing Artin $L$-functions as linear combinations of Epstein zeta functions lies in the existence of a number of relatively rapidly convergent expansions for Epstein zeta functions. These are discussed in Section 4. The particular one we used in our computations is an expansion in terms of incomplete gamma functions. It would be useful to find rapidly convergent expansions of a similar nature for a wider class of Hecke or Artin $L$-functions. Calculations of the values of Epstein zeta functions on the real axis using this or other expansions are described in [21], [23], [26], [27], [32].

The computations are described in Section 5. The fields $K$ considered were $K = \mathbf{Q}(\sqrt{-3}, a^{1/3})$ with $a = 2, 3, 6$, and 12. Here $\mathrm{Gal}(K/\mathbf{Q}) \cong S_3$ and the Artin $L$-functions computed were those attached to the unique irreducible character of degree 2 for $S_3$. These particular Artin $L$-functions can be written in the form $\zeta_L(s)/\zeta(s)$, where $\zeta_L(s)$ is the Dedekind zeta function of the pure cubic field $L = \mathbf{Q}(a^{1/3})$. It is known [36] that such functions have infinitely many zeroes of odd multiplicity on the critical line. For each of these Artin $L$-functions we located all zeroes in the region $0 < \mathrm{Re}(s) < 1$, $|\mathrm{Im}(s)| \leqslant 15$. These zeroes were all on the critical line and were simple.

There are several limitations on the computational methods we used, which restricted us to examining the region $|\mathrm{Im}(s)| \leqslant 15$. They are the difficulty of computing incomplete gamma functions, a certain cancellation effect inherent in the particular Epstein zeta function expansion used, and the use of the argument principle to count zeroes. Section 6 discusses how these limitations could be circumvented so as to carry out more extensive computations.

Since this approach involves Epstein zeta functions, we mention some related results concerning individual Esptein zeta functions. In general, such functions have infinitely many zeroes $\rho$ with $\mathrm{Re}(\rho) > 1$ [11]. On the other hand they have infinitely many zeroes on the critical line, and most of their zeroes $\rho$ have $\mathrm{Re}(\rho)$ close to ½ [20]. There are some special factors which keep zeroes sufficiently close to the real axis (but not on it) on the critical line [29]. We have carried out some preliminary computations for the five Epstein zeta functions involved in the Artin $L$-function corresponding to $\mathbf{Q}(6^{1/3})$, which seem to indicate that zeroes occur off the critical line about as soon as they can be consistent with [29], and that there are a substantial number of zeroes off the critical line. In order to make reasonable guesses concerning the distribution of such zeroes, computations would have to be carried out to much greater heights in the critical strip.

**2. Artin $L$-Functions.** Let $K/k$ be a normal extension of number fields with Galois group $G = \mathrm{Gal}(K/k)$. If $\Psi$ is any character on $G$, the associated Artin $L$-function $L(s, \Psi, K/k)$ is defined for $\mathrm{Re}(s) > 1$ by ([3], [4], [17], [19]):

$$\log L(s, \Psi, K/k) = \sum_{\mathfrak{p}} \sum_{m=1}^{\infty} m^{-1}\Psi(\mathfrak{p}^m)(N_{k/\mathbf{Q}}\,\mathfrak{p})^{-ms}, \tag{2.1}$$

where $\mathfrak{p}$ runs over all prime ideals of $k$ and

$$\Psi(\mathfrak{p}^m) = |I|^{-1} \sum_{\sigma \in I} \Psi(\tau^m \sigma),$$

where $I$ is the inertia group of $\mathfrak{p}$, which is trivial for unramified primes, and $\tau$ is any one of the Frobenius automorphisms corresponding to $\mathfrak{p}$, i.e., any automorphism for which

$$\alpha^{\tau} \equiv \alpha^{N_{k/\mathbf{Q}}\mathfrak{p}} \pmod{\mathfrak{P}}$$

for all algebraic integers $\alpha$ in $K$, with $\mathfrak{P}$ is a fixed prime ideal of $K$ lying over $\mathfrak{p}$.

The Dirichlet series representation (2.1) converges only for $\mathrm{Re}(s) > 1$, but it is known that $L(s, \Psi, K/k)$ can be analytically continued to a function meromorphic in the entire plane [17]. The group representation theory for $G$ is compatible with Artin $L$-functions as follows [17, p. 221]:

PROPOSITION 2.1. (i) *If $\Psi_0$ denotes the identity character of* $\mathrm{Gal}(K/k)$, *then $L(s, \Psi_0, K/k) = \zeta_k(s)$, the Dedekind zeta function of $k$.*

(ii) $L(s, \Psi_1 + \Psi_2, K/k) = L(s, \Psi_1, K/k)L(s, \Psi_2, K/k)$.

(iii) *If $k \subset N \subset K$ and $N/k$ is Galois, then a character $\Psi$ of* $\mathrm{Gal}(K/k)$ *can be viewed as a character of* $\mathrm{Gal}(N/k)$ *and $L(s, \Psi, K/k) = L(s, \Psi, N/k)$.*

(iv) *If $k \subset F \subset K$, $\Psi$ is a character of* $\mathrm{Gal}(K/F)$, *and $\Psi^*$ the induced character*

*on* Gal$(K/k)$, *then* $L(s, \Psi, K/F) = L(s, \Psi^*, K/k)$.

This proposition can be used to exhibit relations between $L$-functions, as in the following example.

COROLLARY 2.2. *Let* Gal$(K/\mathbf{Q}) \cong S_3$ *be the symmetric group on three letters, and let $L$ be any cubic subfield of $K$. Let $\Psi$ denote the unique irreducible character of degree* 2 *on* Gal$(K/\mathbf{Q})$. *Then*

$$L(s, \Psi, K/\mathbf{Q}) = \zeta_L(s)/\zeta(s).$$

*Proof.* The structure of the Artin $L$-functions for the $S_3$ case is worked out in [17, pp. 225–227]. There is a single irreducible character of degree 2. If $L$ is a cubic subfield of $K$, it is fixed by a subgroup $S$ generated by a 2-cycle. If $\chi_0$ denotes the identity character on $S$, then the induced character $\chi_0^* = \Psi_1 + \Psi$, where $\Psi_1$ is the identity character on $G$. Applying Proposition 2.1, we obtain

$$\begin{aligned} \zeta_L(s) &= L(s, \chi_0, K/L) && \text{by (i)} \\ &= L(s, \Psi_1 + \Psi, K/\mathbf{Q}) && \text{by (iv)} \\ &= \zeta(s)L(s, \Psi, K/\mathbf{Q}) && \text{by (i), (ii),} \end{aligned}$$

where $\zeta(s)$ is the ordinary Riemann zeta function. □

The Artin Reciprocity Law is the basis of the relation between Artin $L$-functions and Hecke (abelian) $L$-functions. It can be expressed in the following analytic form [17, p. 221].

PROPOSITION 2.3. *If $\tilde{\chi}$ is a one-dimensional (multiplicative) character on* Gal$(K/k)$, *then there is a conductor modulus $f_\chi$ and a Hecke ray class character $\chi$* (mod $f_\chi$) *such that*

$$L(s, \tilde{\chi}, K/k) = L(s, \chi, k),$$

*where $L(s, \tilde{\chi}, k)$ is the Hecke $L$-function associated to $\tilde{\chi}$.*

Using this result together with group representation theory, it can be shown that every Artin $L$-function can be expressed as a product of Hecke $L$-functions raised to (positive or negative) integer powers [17, Theorem 7ff.], [19, p. 13].

**3. Ring Class Fields and Ring Class $L$-Functions.** In this section we will show that certain Artin $L$-functions attached to irreducible characters of degree greater than one are linear combinations of Epstein zeta functions of positive definite binary quadratic forms, which will be referred to as EZF's. This result is proved in three steps. The first step is to show that some of these Artin $L$-functions are Hecke $L$-functions; this involves only group representation theory. The second step shows that certain Hecke $L$-functions, the ring class $L$-functions of a quadratic field, are linear combinations of EZF's. The third step is to determine extra conditions necessary for Artin $L$-functions to be ring class Hecke $L$-functions. Together these results yield the main result of this section, Theorem 3.7.

We consider Galois groups $G$ which have an abelian subgroup $A$ of index 2. (Since $[G:A] = 2$, $A$ is normal in $G$.) The representation theory for such a group is known [10, p. 341].

PROPOSITION 3.1. *Let $G$ be a group with an abelian subgroup of index* 2. *Then all irreducible characters of $G$ are of degree* 1 *or* 2, *and each degree* 2 *irreducible character is induced from some degree* 1 *character of $A$.*

We can now prove:

LEMMA 3.2. *Let* $\mathrm{Gal}(K/\mathbf{Q})$ *have an abelian subgroup $A$ of index* 2, *and let* $\mathbf{Q}(\sqrt{d})$ *be the quadratic field fixed by $A$. If $\Psi$ is an irreducible character of degree* 2 *on* $\mathrm{Gal}(K/\mathbf{Q})$, *then there is a Hecke ray class character $\chi$ for some conductor modulus $f_\chi$ on* $\mathbf{Q}(\sqrt{d})$ *such that*

$$L(s, \Psi, K/\mathbf{Q}) = L(s, \chi, \mathbf{Q}(\sqrt{d})).$$

*Proof.* By Proposition 3.1 there is a 1-dimensional character $\chi$ on $A = \mathrm{Gal}(K/\mathbf{Q}(\sqrt{d}))$ such that $\chi^* = \Psi$. By Proposition 2.1(iv)

$$L(s, \Psi, K/\mathbf{Q}) = L(s, \chi, K/\mathbf{Q}(\sqrt{d})).$$

Since $\chi$ is 1-dimensional, by Proposition 2.3

$$L(s, \chi, K/\mathbf{Q}(\sqrt{d})) = L(s, \tilde{\chi}, \mathbf{Q}(\sqrt{d}))$$

for some Hecke ray-class character $\chi \pmod{f_\chi}$. □

We now turn to Epstein zeta functions. The Epstein zeta function $Z(s, Q)$ associated to a positive definite binary quadratic form $Q$ is defined by

$$Z(s, Q) = \sum_{(x,y)\neq(0,0)} Q(x, y)^{-s},$$

where $(x, y)$ runs over all pairs of integers other than $(0, 0)$.

The particular Hecke $L$-functions, which we will show are linear combinations of Epstein zeta functions, are special kinds of $L$-functions attached to complex quadratic fields called ring class $L$-functions.

*Definition.* Let $\chi$ be a Hecke character attached to a complex quadratic field $\mathbf{Q}(\sqrt{d})$. $\chi$ is called a *ring class character* $(\mathrm{mod}\, f)$ if there exists an integer $f$ such that $\chi((\alpha)) = 1$ for all principal ideals $(\alpha)$ in $\mathbf{Q}(\sqrt{d})$ for which $\alpha \equiv h \pmod{f}$ for some rational integer $h$ with $(\alpha, h) = 1$. The corresponding Hecke $L$-function $L(s, \chi, \mathbf{Q}(\sqrt{d}))$ is called a *ring class $L$-function* $(\mathrm{mod}\, f)$.

LEMMA 3.3. *Let $\chi$ be a ring class character* $(\mathrm{mod}\, f)$ *attached to a complex quadratic field* $\mathbf{Q}(\sqrt{d})$. *Then there exists a finite set of binary quadratic forms $Q_1, \dots, Q_n$ of discriminant $df^2$ and coefficients $c(\chi, Q_i)$ such that*

$$L(s, \chi, \mathbf{Q}(\sqrt{d})) = \sum_{j=1}^{n} c(\chi, Q_j) Z(s, Q_j).$$

*Proof.* Let $I_f$ denote the fractional ideals of $\mathbf{Q}(\sqrt{d})$ prime to $f$ and let

$$P_f = \{(\alpha)\colon \alpha \equiv h \pmod f \text{ for some } h \in \mathbf{Z} \text{ with } (h, f) = 1\}$$

denote the *principal ring class* (mod $f$). Since $\chi$ is constant on $P_f$ by hypothesis, $\chi$ is constant on ring classes and so

$$L(s, \chi, \mathbf{Q}(\sqrt{d})) = \sum_{[A] \in I_f/P_f} \chi(A) \sum_{B \in [A]} (NB)^{-s}.$$

But it is known [30, Lemma 27] that for each ring class $[A]$

$$\sum_{B \in [A]} (NB)^{-s} = \frac{1}{W(df^2)} Z(s, Q)$$

for some binary quadratic form $Q$ of discriminants $df^2$, where

$$W(df^2) = \begin{cases} 6, & df^2 = -3, \\ 4, & df^2 = -4, \\ 2, & \text{otherwise.} \end{cases}$$

This completes the proof. □

The *ring class field* (mod $f$) is that class field over $\mathbf{Q}(\sqrt{d})$ for which the prime ideals that split completely over $\mathbf{Q}(\sqrt{d})$ are exactly those in the principal ring class (mod $f$). This field is guaranteed to exist by the fundamental theorem of class field theory, and we denote it by $K_{df^2}$.

PROPOSITION 3.4. *Let $K$ be normal over $\mathbf{Q}$ and suppose $\mathbf{Q}(\sqrt{d}) \subseteq K \subseteq K_{df^2}$ for some ring class field. Then*

$$\zeta_k(s) = \prod_{\chi} L(s, \chi, \mathbf{Q}(\sqrt{d})), \tag{3.1}$$

*where the product runs over a certain set of primitive Hecke characters $\chi$ which are ring class characters* (mod $f$).

*Proof.* This follows from [17, Theorem 6]. □

In particular by Lemma 3.3 this gives a decomposition of the Dedekind zeta function of such fields into products of linear combinations of EZF's.

Brückner [6, Theorem 8] gave a Galois-theoretic characterization of ring class fields over both real and complex quadratic fields. From this we can easily derive a criterion for the complex quadratic case.

PROPOSITION 3.5. *The following are equivalent*:

(i) *$K$ is normal over $\mathbf{Q}$ and* $\mathrm{Gal}(K/\mathbf{Q})$ *has an abelian subgroup of index 2 whose fixed field is a complex quadratic field $\mathbf{Q}(\sqrt{d})$. Furthermore there is a $\sigma \in G-A$ such that*

$$\sigma a \sigma^{-1} = a^{-1} \quad \textit{for all } a \in A. \tag{3.2}$$

(ii) *$K$ is normal over $\mathbf{Q}$ and there is a ring class field $K_{df^2}$ with $\mathbf{Q}(\sqrt{d}) \subseteq K \subseteq K_{df^2}$, where $\mathbf{Q}(\sqrt{d})$ is complex.*

*Proof.* To apply Brückner's result [6, Theorem 8], we need only check that

$\sigma^2 = e$. If $\tau$ denotes complex conjugation, then $\tau \in G-A$; hence $\tau = \sigma a$ for some $a \in A$, and $e = \tau^2 = (\sigma a)^2 = \sigma a a^{-1}\sigma = \sigma^2$. □

Applying Lemma 3.3 and Proposition 3.4 yields the following.

COROLLARY 3.6. *If $K$ is normal over* $\mathbf{Q}$, $\mathrm{Gal}(K/\mathbf{Q})$ *has an abelian subgroup $A$ of index* 2 *whose fixed field* $\mathbf{Q}(\sqrt{d})$ *is complex, and there exists an element $\sigma \in G-A$ with*

$$\sigma a \sigma^{-1} = a^{-1} \quad \textit{for all } a \in A,$$

*then $\zeta_K(s)$ factors into a product of linear combinations of Epstein zeta functions.*

For example, this includes all totally complex $K$ with $\mathrm{Gal}(K/\mathbf{Q}) \cong \mathbf{Z}_2 \oplus \mathbf{Z}_2$ or the symmetric group $S_3$, since the totally complex hypothesis guarantees that the appropriate quadratic subfield is complex.

THEOREM 3.7. *Suppose that $K$ is normal over* $\mathbf{Q}$, $\mathrm{Gal}(K/\mathbf{Q})$ *has an abelian subgroup of index* 2 *whose fixed field* $\mathbf{Q}(\sqrt{d})$ *is complex, and that there is a $\sigma \in G-A$ with*

$$\sigma a \sigma^{-1} = a^{-1} \quad \textit{for all } a \in A.$$

*If $\psi$ is an irreducible character of degree* 2 *on* $\mathrm{Gal}(K/\mathbf{Q})$, *then there are quadratic forms $Q_1, \ldots, Q_n$ of discriminant $df^2$ and coefficients $c(\psi, Q_j)$ such that the Artin $L$-function $L(s, \psi, K/\mathbf{Q})$ is given by*

$$L(s, \psi, K/\mathbf{Q}) = \sum_{j=1}^{n} c(\psi, Q_j) Z(s, Q_j).$$

*Proof.* By Lemma 3.2

$$L(s, \psi, K/\mathbf{Q}) = L(s, \chi, K/\mathbf{Q}(\sqrt{d}))$$

for a certain Hecke $L$-function of modulus $f_\chi$ on $\mathbf{Q}(\sqrt{d})$. Take $f = Nf_\chi$ a rational integer. By Proposition 3.5, $\mathbf{Q}(\sqrt{d}) \subseteq K \subseteq K_{df^2}$. Hence, all principal prime ideals $(\alpha)$ in the principal ring class (mod $f$) split completely in $K$, so that the Artin symbol

$$\left[\frac{K/\mathbf{Q}(\sqrt{d})}{(\alpha)}\right] = e.$$

Hence,

$$\chi((\alpha)) = \chi\left(\left[\frac{K/\mathbf{Q}(\sqrt{d})}{(\alpha)}\right]\right) = 1,$$

since $\chi$ is multiplicative. Since $\chi$ is constant on ray classes (mod $f$), and there is a prime ideal $(\alpha)$ in each ray class in the principal ring class (mod $f$), $\chi \equiv 1$ on the principal ring class (mod $f$) and so is a ring class character. By Lemma 3.3 we are done. □

Although this theorem guarantees the existence of representations of these Artin $L$-functions as linear combinations of EZF's, it does not provide a method for finding these representations. However, if we are willing to forego knowing to which field $K$

these Artin $L$-functions are attached, we can proceed by finding instead the representations for all ring class $L$-functions over complex quadratic fields. To list all ring class $L$-functions (mod $f$) over $\mathbf{Q}(\sqrt{d})$ we first find a set $Q_1, \ldots, Q_n$ of inequivalent forms of discriminant $df^2$ (for example, reduced forms) and next determine the multiplication table of the group of equivalence classes of forms under (Gaussian) composition. Such computations may be done quite efficiently (cf. [25], [26]). As $\chi$ runs over all characters on this group,

$$\frac{1}{W(df^2)} \sum_{i=1}^{n} \chi(Q_i) Z(s, Q_i)$$

runs over all the ring class $L$-functions (mod $f$) over $\mathbf{Q}(\sqrt{d})$.

In the special case $K = \mathbf{Q}(\sqrt{-3}, a^{1/3})$, where $a$ is not a perfect cube, $\mathrm{Gal}(K/\mathbf{Q}) \cong S_3$ and the Artin $L$-function attached to the unique irreducible character $\Psi$ of degree 2 is given by

$$L(s, \Psi, K/\mathbf{Q}) = \frac{\zeta_L(s)}{\zeta(s)},$$

where $L = \mathbf{Q}(a^{1/3})$ by Corollary 2.2. The EZF decomposition for $\zeta_L(s)/\zeta(s)$ can be worked out using the decomposition law in cubic fields, and this was done by Dedekind [14] in 1879. (Alternate methods available in this case are considered in [7], [9].) If we set

$$L_a(s) = L(s, \Psi, \mathbf{Q}(\sqrt{-3}, a^{1/3})/\mathbf{Q}) = \frac{\zeta_{\mathbf{Q}(a^{1/3})}(s)}{\zeta_{\mathbf{Q}}(s)}, \tag{3.3}$$

then Dedekind showed [14, p. 107] that

$$L_2(s) = \tfrac{1}{2} Z(s, x^2 + 27y^2) - \tfrac{1}{2} Z(s, 4x^2 + 2xy + 7y^2), \tag{3.4a}$$

$$L_3(s) = \tfrac{1}{2} Z(s, x^2 + xy + 61y^2) - \tfrac{1}{2} Z(s, 7x^2 + 3xy + 9y^2), \tag{3.4b}$$

$$\begin{aligned} L_6(s) = {} & \tfrac{1}{2} Z(s, x^2 + 243y^2) + Z(s, 7x^2 + 6xy + 36y^2) \\ & - \tfrac{1}{2} Z(s, 9x^2 + 6xy + 28y^2) - \tfrac{1}{2} Z(s, 4x^2 + 2xy + 61y^2) \\ & - \tfrac{1}{2} Z(s, 13x^2 + 4xy + 19y^2), \end{aligned} \tag{3.4c}$$

$$\begin{aligned} L_{12}(s) = {} & \tfrac{1}{2} Z(s, x^2 + 243y^2) - \tfrac{1}{2} Z(s, 7x^2 + 6xy + 36y^2) \\ & - Z(s, 9x^2 + 6xy + 28y^2) - \tfrac{1}{2} Z(s, 4x^2 + 2xy + 61y^2) \\ & + Z(s, 13x^2 + 4xy + 19y^2). \end{aligned} \tag{3.4d}$$

These are the particular Artin $L$-functions whose zeroes we computed.

**4. Epstein Zeta Functions.** This section briefly describes two rapidly convergent expansions that are available for Epstein zeta functions associated to positive definite binary quadratic forms. In fact, expansions exist for Epstein zeta functions attached to positive definite quadratic forms in several variables, cf. [31], [32].

The first is a Fourier expansion in terms of incomplete $K$-Bessel functions of the second kind, based on the fact that $Z(s, Q)$ is a nonanalytic Eisenstein series for $GL(2, \mathbf{R})$. (See [5], [8], [31].)

PROPOSITION 4.1. *If $Q(x, y) = Ax^2 + Bxy + Cy^2$ with $C > 0$, then*

$$Z(s, Q) = C^{-s}\zeta(2s) + \pi^{1/2}C^{-1/2}T^{1/2-s}\frac{\Gamma(s-1/2)}{\Gamma(s)}\zeta(2s-1)$$
$$+ (4\pi)^{1/2}C^{-1/2}T^{1/2-s}\frac{1}{\Gamma(s)}H(s),$$

*where*

$$T = -(4C)^{-1}(B^2 - 4AC) > 0,$$

$$H(s) = \sum_{m\geq 1}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}(\pi T^{1/2}C^{-1/2}m|n|)^{s-1/2}m^{1-2s}$$
$$\cdot \cos\left(\frac{\pi B}{C}mn\right)K_{1/2-s}(2\pi T^{1/2}C^{-1/2}m|n|)$$

*and*

$$\hat{K}_s(z) = \frac{1}{2}\int_0^\infty \exp\left\{-\frac{z}{2}\left(u + \frac{1}{u}\right)\right\}u^{s-1}\,du$$

*defined for* $|\operatorname{Arg} z| < \pi/2$ *is the modified Bessel function of the second kind.*

The second expansion is in terms of incomplete gamma functions, and is based on the fact that $Z(s, Q)$ is a Mellin transform of a theta function [32, Theorem 2].

PROPOSITION 4.2. *Let $Q(x, y) = Ax^2 + Bxy + Cy^2$, with determinant $D = AC - B^2/4 > 0$, and let $\delta \in \mathbf{C}$ have* $\operatorname{Re}(\delta) > 0$. *Set*

$$\Lambda(s, Q) = (\pi\delta)^{-s}\Gamma(s)Z(s, Q) = \pi^{-s}\Gamma(s)Z(s, \delta Q).$$

*Then*

$$\Lambda(s, Q) = -\frac{1}{s} - \frac{(\delta^2 D)^{-1/2}}{1-s}$$
$$+ \sum_{\substack{x,y\\ (x,y)\neq(0,0)}}\{G(s, \pi\delta Q(x, y)) + (\delta^2 D)^{-1/2}G(1-s, \pi\delta^{-1}D^{-1}Q(x, y))\}, \tag{4.1}$$

*where*

$$G(s, \alpha) = \alpha^{-s}\Gamma(s, \alpha) = \int_1^\infty t^{s-1}e^{-\alpha t}\,dt, \tag{4.2}$$

*valid for* $\operatorname{Re}(\alpha) > 0$.

Here $\Gamma(s, \alpha)$ is the incomplete gamma function, which is usually defined by the integral

$$\Gamma(s, \alpha) = \int_\alpha^\infty t^{s-1}e^{-t}\,dt, \tag{4.3}$$

where the contour of integration is required to stay inside the region $|\arg t| < \pi/2$.

The usefulness of these two expansions depends on the availability of algorithms to compute the modified Bessel function of the second kind and the incomplete gamma function, respectively. No completely satisfactory algorithms are known to us, but the incomplete gamma function currently seems easier to deal with. This dictated our choice of the expansion (4.1) for computational purposes. We set the parameter $\delta = D^{-1/2}$, so that $\det(\delta Q) = 1$, which simplifies (4.1) somewhat. We emphasize that (4.1) is valid for all $\delta$ with $\mathrm{Re}(\delta) > 0$. Although we did not make use of this extra generality in our computations, it may be useful in using this approach for more extensive computations, (cf. Section 6).

**5. Computational Results.** We determined the approximate location of all the complex zeroes $\rho = \sigma + it$ of the four Artin $L$-functions $L_a(s)$ given by (3.4) in the region $0 < \sigma < 1$, $|t| < 15$. According to (3.3) these Artin $L$-functions are equal to $\zeta_L(s)/\zeta(s)$ for $L = \mathbf{Q}(a^{1/3})$, where $a = 2, 3, 6$, and 12. The results appear in Table 1. All the zeroes are simple and are on the critical line $\sigma = ½$. Furthermore, the zeroes are symmetric about the real axis, so that only the ordinates $t$ of the zeroes with $t > 0$ are listed in Table 1. (As a by-product of the computation, two zeroes with $|t| > 15$ are included.) The values given are accurate to one unit in the last decimal place. We note that no two of these $L$-functions have a common zero in the region examined, and none of these zeroes coincides with the smallest nontrivial zero $\rho \cong 0.5 + 14.1347i$ of the Riemann zeta function. The value 5.9999 in the column $a = 12$ is definitely not the integer 6.

The computation took place in two stages. The first consisted of locating zeroes on the critical line ½. To do this, we computed $\Lambda_a(s) = 2^{-s}D_a^{-s/2}\Gamma(s)L_a(s)$, where $D_a = 3^3, ¼ \cdot 3^5, 3^5, 3^5$ for $a = 2, 3, 6, 12$, respectively, using the expansion of Theorem 4.2, at a sequence of points on the critical line in the region $0 \leqslant t \leqslant T$, where $T$ was chosen suitably. $\Lambda_a(s)$ is real on the critical line (because it satisfies the functional equation $\Lambda_a(s) = \Lambda_a(1-s)$) and, hence, the number of sign changes of $\Lambda_a(s)$ detected on the critical line is a lower bound for the number of zeroes of $L_a(s)$ on the critical line in that region. The second stage consisted of an application of the argument principle to determine the total number of zeroes (counting multiplicity) in the region $0 < \mathrm{Re}(s) < 1$, $|\mathrm{Im}(s)| \leqslant T$. Since the second number was twice the first for the four functions we computed, we concluded that all the zeroes were simple and on the critical line. In the remainder of this section we describe the computations in more detail, pointing out three limitations of our approach that led us to choose $T \leqslant 16$.

Both stages of this computation require computing $L_a(s)$ for complex values of $s$. We used the EZF expansions (3.4) for $L_a(s)$. In those cases $D_a$ is exactly the determinant of all the corresponding quadratic forms in the expansions (3.4), and the gamma factors in $\Lambda_a(s)$ give rise to exactly the corresponding left-hand sides of (4.1) in Proposition 4.2, provided we choose $\delta = (D_a)^{-1/2}$. In that case (4.1) becomes

$$(5.1)\quad \Lambda(s, Q) = \frac{1}{s(s-1)} + \sum_{\substack{x,y \\ (x,y)\neq(0,0)}} \{G(s, \pi(D_a)^{-1/2}Q(x,y)) + G(1-s, \pi(D_a)^{-1/2}Q(x,y))\}.$$

TABLE 1

*Zeros $\rho = \sigma + it$ of $\zeta_L(s)/\zeta(s)$, $L = \mathbf{Q}(a^{1/3})$ with $0 < t < 15$. All these zeroes have $\sigma = \frac{1}{2}$, so only the values of $t$ are given.*

| $a = 2$ | $a = 3$ | $a = 6$ | $a = 12$ |
|---|---|---|---|
| 2.8216 | 2.2936 | 1.5654 | 1.3371 |
| 4.5401 | 3.5288 | 2.6232 | 3.1589 |
| 6.1435 | 5.2442 | 4.2924 | 3.6358 |
| 7.2066 | 6.5394 | 4.8317 | 5.0959 |
| 8.6454 | 7.0925 | 5.7575 | 5.9999 |
| 9.7644 | 8.3878 | 7.0211 | 6.7446 |
| 10.6634 | 9.6447 | 7.7836 | 7.6588 |
| 12.1139 | 10.4387 | 8.4467 | 8.7441 |
| 12.8115 | 11.3704 | 9.2757 | 9.2822 |
| 13.9284 | 12.4669 | 10.3991 | 10.2246 |
| 14.9951 | 12.8922 | 11.0254 | 10.9676 |
| | 14.3281 | 11.6787 | 11.7582 |
| | (15.0741) | 12.4568 | 12.6027 |
| | | 13.3031 | 13.4468 |
| | | 14.2951 | 13.7589 |
| | | 14.8421 | 14.8129 |
| | | (15.0658) | |

From this equation it is obvious that $\Lambda(s, Q)$ is real on the critical line $\mathrm{Re}(s) = \frac{1}{2}$ and satisfies the functional equation $\Lambda(s, Q) = \Lambda(1 - s, Q)$. From (3.4), $\Lambda_a(s)$ also is real on the critical line and $\Lambda_a(s) = \Lambda_a(1 - s)$.

For the actual computation we combined the incomplete gamma function expansions (5.1) for the various EZF's involved in $\Lambda_a(s)$ to obtain

$$\Lambda_a(s) = \sum_{u \in U} c_u \{G(s, \pi D_a^{-1/2} u) + G(1 - s, \pi D_a^{-1/2} u)\}, \tag{5.2}$$

where $U$ was the set of values taken by the quadratic forms involved and the $c_u$ are coefficients computed using (3.4). (Note that the $1/s(s - 1)$ terms cancelled out of this expression.) For $0 < \mathrm{Re}(s) < 1$ and $\alpha \geqslant 1$ we have the trivial bound

$$|\Gamma(s, \alpha)| \leqslant \int_\alpha^\infty e^{-u}\, du = e^{-\alpha}. \tag{5.3}$$

In the computation we used (5.3) and similar estimates to truncate the sum in (5.2) at a point where the remainder was bounded by absolute value by $2 \times 10^{-13}$ in the region

$$\{s\colon 0 \leqslant \mathrm{Re}(s) \leqslant 1,\ 0 \leqslant \mathrm{Im}(s) \leqslant 16 \text{ and } -\tfrac{1}{2} \leqslant \mathrm{Re}(s) \leqslant 2,\ 15 \leqslant \mathrm{Im}(s) \leqslant 16\}. \tag{5.4}$$

In the case $a = 2$ and $a = 12$, this involved taking 8 and 20 values of $u \in U$, respectively.

The first limitation of this approach is inherent in expansion (5.1), more generally in choosing a fixed value of $\delta$ in Proposition 4.2 to use for all $s$. If $\alpha$ is held fixed and $s = \frac{1}{2} + it$ with $t \longrightarrow \infty$, then the incomplete gamma function $\Gamma(s, \alpha)$ decreases like $|s|^{-1}$. On the other hand, $\Gamma(s)$ decreases like $\exp(-|s| \log|s|)$ for $s = \frac{1}{2} + it$ with $t \longrightarrow \infty$ by Stirling's formula, while $L_a(\frac{1}{2} + it)$ is known to grow relatively slowly. Hence, for large $t$ the expansion (5.1) involves relatively large terms (compared to the value of $\Lambda_a(s)$) which must be cancelling each other. Since the number of significant figures is limited by the largest element occurring in the course of a summation, expansion (5.1) is useful on a typical machine with double precision arithmetic only for $T \leqslant 20$, say.

The second limitation we encountered was in computing the incomplete gamma function $\Gamma(s, \alpha)$. There are a large number of algorithms available for computing $\Gamma(s, \alpha)$ when $s$ and $\alpha$ are both real [15]. If $s$ is complex, we do not know of any very satisfactory algorithm. Numerical integration as used in [35] does not seem very efficient when $\operatorname{Im}(s)$ exceeds 10. The continued fraction method (see [21], [33]) seems very promising, but has not been proved to converge numerically for $s$ off the real axis. Therefore, we have used the following expansion [15]

$$\Gamma(s, \alpha) = \Gamma(s) - \alpha^s e^{-\alpha} \sum_{k=0}^{\infty} \frac{\alpha^k}{s(s+1)\cdots(s+k)}.$$

This formula is easily verified. Using (4.2) and repeated integrations by parts:

$$\begin{aligned} \Gamma(s) - \Gamma(s, \alpha) &= \int_0^\alpha e^{-u} u^{s-1}\, du = \frac{u^s e^{-u}}{s}\bigg|_0^\alpha + \frac{1}{s}\int_0^\infty e^{-u} u^s\, du \\ &= \alpha^s e^{-\alpha} \sum_{k=0}^{N} \frac{\alpha^k}{s(s+1)\cdots(s+k)} \\ &\quad + \frac{1}{s(s+1)\cdots(s+N)} \int_0^\alpha e^{-u} u^{s+N}\, du. \end{aligned} \tag{5.5}$$

In using this expansion, $\Gamma(s)$ was computed by estimating $\Gamma(s + 15)$ by Stirling's formula [1, Section 6.142] and then computing $\Gamma(s)$ from it via the recurrence $s\Gamma(s) = \Gamma(s + 1)$. A value of $N$ on the right side of (5.5) was chosen for which the remaining integral could be bounded (crudely) by $e^{-\alpha} \cdot 10^{-17}$. (This involved $N$ on the order of 10 to 50 for the values of $\alpha$ and $s$ considered.)

The unsatisfactory feature of expansion (5.5) as it stands is that if $\alpha$ is large compared to $|s|$ then the maximal term in the sum in (5.5) will be large compared to the final answer, and substantial cancellation must occur when $\operatorname{Im}(s)$ is large. This limits the number of significant figures attainable in the computation.

These considerations sufficed to compute $\Lambda_a(s)$ to accuracy $10^{-12}$ in the region (5.4). We first computed $\Lambda_a(0.5 + 0.1ni)$ for $0 \leqslant n \leqslant 160$ which enabled us to

separate all the zeroes in Table 1. By successive approximation each of the zeroes so separated was then located to four decimal places.

We now describe the argument principle computation. The argument principle asserts the number of zeroes of an analytic function inside a contour (assuming none on the contour) is the change in the argument around the contour divided by $2\pi$. We consider the rectangular contour with vertices at $2 + iT$, $-2 + iT$, $-2 - iT$, $2 - iT$. Using the symmetries of $\Lambda_a(s)$ (real on real axis, $\Lambda_a(s) = \Lambda_a(1 - s)$) we conclude that the number of zeroes of $\Lambda_a(s)$ inside this contour is $2/\pi$ times the change in argument of $\Lambda_a(s)$ as $s$ goes vertically from $s = 2$ to $s = 2 + iT$, and then horizontally to $s = \frac{1}{2} + iT$. The change in argument of $\Lambda_a(s)$ from $s = 2$ to $s = 2 + iT$ is easy to estimate. The argument change of a product is the sum of the argument changes of its factors. The change due to $\pi^{-s}D_a^{-s/2}\Gamma(s)$ is easy to compute. This is essentially the entire change, as $L_a(s)$ can be bounded on the vertical segment using its Euler product (for all $t$) by

$$|\log L_a(2 + it)| \leqslant 2 \log \zeta(2) < 0.996,$$

hence its change in argument is $\arg L_a(2 + iT)$.

The computation of the argument change of $\Lambda_a(s)$ on the horizontal segment from $2 + iT$ to $\frac{1}{2} + iT$ is much harder. We resorted to computing $\Lambda_a(s)$ at a large set of points on this line (on the order of 50), along with its first three derivatives, which were computed using a differentiated form of (5.6). We obtained upper bounds on the fourth derivative of $\Lambda_a(s)|$ valid in the region $15 \leqslant \mathrm{Im}(s) \leqslant 16, -1 \leqslant \mathrm{Re}(s) \leqslant 2$, and could then show $\Lambda_a(s)$ could not change its argument by as much as $\pi$ between these points. The total argument change was then the sum of the differences in arguments at consecutive points. The values of $T$ for each $\Lambda_a(s)$ were chosen between 15 and 16 to be roughly halfway between the ordinates of zeroes previously located on the critical line, in order to avoid points where $\Lambda_a(s)$ is small. This accounts for the two zeroes with $t > 15$ included in Table 1.

The third limitation of this method is the use of the argument principle in this form. As $T$ becomes larger, the number of points at which $\Lambda_a(s)$ must be evaluated on the segment $2 + iT$ to $\frac{1}{2} + iT$ seems to become prohibitive. However, see the next section.

All of the computations were carried out in double precision on a Honeywell 6000 computer. This machine has at least 17 significant figures.

**6. Concluding Remarks.** It would be useful to be able to do computations of these functions to much greater heights in the critical strip. We indicate how the three limitations discussed in the last section may potentially be circumvented.

The first limitation is the large size of some of the individual terms in the incomplete gamma function expansions compared to the final answer. This is avoidable by choosing the parameter $\delta$ in Proposition 4.1 to have appropriate complex values. Of course, this method is useful only if the resulting incomplete gamma functions (which have both arguments complex) can be computed.

The second limitation is the difficulty of computing the incomplete gamma function. The continued fraction algorithm for $\Gamma(s, \alpha)$ developed by R. Terras (see [21], [33]) has been proved to converge numerically for $s$ real, $\alpha > 0$ real. We have observed empirically that this algorithm converges to fifteen significant figure accuracy for $\alpha > 0$ real and $s$ complex on the critical line, with the convergence becoming (apparently) more rapid as $\mathrm{Im}(s)$ increases. It seems likely that this algorithm can in fact be used to compute $\Gamma(s, \alpha)$ to fifteen significant figures for a wide range of complex $s$ and $\alpha$; it would be useful to prove it is numerically stable in certain complex domains.

If these two limitations are overcome, then we could compute $\Lambda_a(s)$ with $\mathrm{Im}(s)$ large. In particular, we could get lower bounds for the number of zeroes on the critical line to some large height.

The third difficulty is the number of interpolation points necessary to apply the argument principle, to prove that all the complex zeroes are on the critical line and are simple (if this is so). Instead of doing this, one can generalize the elegant method of Turing (see [18], [22]) which has been used to prove that all the zeroes of the Riemann zeta function $\zeta(s)$ are simple and on the critical line, up to a great height. This method uses only values of $\zeta(\frac{1}{2} + it)$ for various $t$. However, the Turing method, even if generalized to our case, would not have helped us as it requires computing values on the critical line substantially higher than the region in which one hopes to prove all zeroes are simple and on the critical line.

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