On the Zeros of the Riemann Zeta Function in the Critical Strip

By Richard P. Brent

Abstract. We describe a computation which shows that the Riemann zeta function \( \zeta(s) \) has exactly 75,000,000 zeros of the form \( \sigma + it \) in the region \( 0 < t < 32,585,736.4 \); all these zeros are simple and lie on the line \( \sigma = \frac{1}{2} \). (A similar result for the first 3,500,000 zeros was established by Rosser, Yohe and Schoenfeld.) Counts of the number of Gram blocks of various types and the number of failures of "Rosser's rule" are given.

1. Introduction. The Riemann zeta function \( \zeta(s) \) is the analytic function of \( s = \sigma + it \) defined by

\[
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}
\]

for \( \sigma > 1 \), and by analytic continuation for \( \sigma < 1, s \neq 1 \). Apart from "trivial" zeros at the negative even integers, all zeros of \( \zeta(s) \) lie in the critical strip \( 0 < \sigma < 1 \). The Riemann hypothesis is the conjecture [22] that all nontrivial zeros of \( \xi(s) \) lie on the critical line \( \sigma = \frac{1}{2} \). For the number-theoretic significance of the Riemann hypothesis see, for example, Edwards [6] or Ingham [10].

Since \( \xi(s) = \overline{\xi(s)} \), we need only consider zeros \( \rho_j = \sigma_j + it_j \) with \( t_j > 0 \). We assume that the zeros \( \rho_j \) are counted according to their multiplicities and ordered so that \( 0 < t_j \leq t_{j+1} \) (and \( \sigma_j \leq \sigma_{j+1} \) if \( t_j = t_{j+1} \)) for \( j \geq 1 \). By "the first \( n \) zeros of \( \xi(s) \)" we mean \( \rho_1, \ldots, \rho_n \). For brevity we let \( H(n) \) denote the statement that the first \( n \) zeros of \( \xi(s) \) are simple and lie on the critical line. Thus, \( H(n) \) holds for arbitrarily large \( n \) if and only if the Riemann hypothesis is true and all zeros of \( \xi(s) \) are simple.

In the era of hand computation, Gram [7], Backlund [2], Hutchinson [9], and Titchmarsh and Comrie [26] established \( H(10), H(79), H(138) \) and \( H(1,041) \), respectively. For a description of these computations see Edwards [6].

D. H. Lehmer [13], [14] performed the first extensive computation of zeros of \( \xi(s) \) on a digital computer and established \( H(25,000) \). Using similar methods, Meller [16], Lehman [11], and Rosser, Yohe and Schoenfeld [23] established \( H(35,337), H(250,000) \), and \( H(3,500,000) \), respectively.

Using essentially the method introduced by Lehmer, we have established \( H(75,000,001) \). Moreover, there are precisely 75,000,000 zeros with \( 0 < t_j < 32,585,736.4 \). The computational method is outlined in Section 4, and additional
details are given in Section 5. In Section 6 the results are summarized and various statistics regarding the distribution of the zeros are tabulated. Preliminary results are given in Sections 2 and 3.

2. Properties of $\xi(s)$. In this section we summarize some well-known properties of $\xi(s)$ which form the basis for the computational method described in Section 4.

2.1. The Functional Equation for $\xi(s)$. $\xi(s)$ satisfies a functional equation which may be written in the form

$$\xi(s) = \xi(1 - s),$$

where

$$\xi(s) = \pi^{-s/2} \Gamma(s/2) \xi(s).$$

It follows that, if

$$(2.1) \quad \theta(t) = \arg \left[ \pi^{-s/2} \Gamma(s/2) e^{i\pi t} \right] = I \left[ \ln \Gamma(s/2) + i\pi t \right] - \frac{1}{2} \ln \pi,$$

then

$$(2.2) \quad Z(t) = \exp \left[ i\theta(t) \right] \xi(s/2 + it)$$

is real for real $t$. Thus, simple zeros of $\xi(s)$ on the critical line can be located by finding changes of sign of $Z(t)$. (The first few zeros of $Z(t)$ are $t_1 = 14.1347, t_2 = 21.0220, t_3 = 25.0109, \ldots$; see Haselgrove and Miller [8].)

2.2. The Asymptotic Expansion for $\theta(t)$. From (2.1) and Stirling's formula for $\ln \Gamma(s/2)$, we obtain the following asymptotic expansion for the phase $\theta(t)$:

$$(2.3) \quad \theta(t) = \frac{1}{2} \ln \left( \frac{t}{2\pi} \right) - \frac{\pi}{8} + \sum_{k=1}^{n} \frac{B_{2k}(1 - 2^{1-2k})}{4k(2k - 1)} t^{1-2k} + r_n(t),$$

where $B_2 = 1/6, B_4 = -1/30, \ldots$ are Bernoulli numbers, and

$$|r_n(t)| < \frac{(2n)!}{(2\pi)^{2n+2} t^{2n+1}} + \exp(-\pi t)$$

for all $t > 0$ and $n \geq 0$.

$\theta(t)$ has a minimum of approximately $-3.53$ near $t = 2\pi$, and is monotonic increasing for $t \geq 7$. For $m \geq -1$, we define the $m$th Gram point $g_m$ to be the unique solution in $[7, \infty)$ of

$$(2.4) \quad \theta(g_m) = mn.$$

Thus, $g_{-1} = 9.6669, g_0 = 17.8456, g_1 = 23.1703, \ldots$.

2.3. The Euler-Maclaurin Formula for $\xi(s)$. $\xi(s)$ may be calculated to any desired accuracy by taking $m$ and $n$ large enough in the Euler-Maclaurin formula

$$(2.5) \quad \xi(s) = \sum_{j=1}^{n-1} j^{-s} + \frac{1}{2} n^{-s} + \frac{n^{1-s}}{s-1} + \sum_{k=1}^{m} T_{k,n}(s) + E_{m,n}(s),$$

where

$$T_{k,n}(s) = \frac{B_{2k}}{(2k)!} \frac{n^{1-s-2k}}{s-1} \prod_{j=0}^{k-2} \left( s+j \right)$$
and

$$|E_{m,n}(s)| < |T_{m+1,n}(s)(s + 2m + 1)/(s + 2m + 1)|$$

for all $m > 0$, $n > 1$, and $s = \Re(s) > -(2m + 1)$.

If (2.5) is used to obtain $\zeta(1/2 + it)$ to within a specified absolute tolerance, then it is necessary to take $n \geq t/(2\pi)$. It is also sufficient to take $n = O(t)$ and $m = O(t)$. Thus, the computational work required is roughly proportional to $t$.

2.4. The Riemann-Siegel Formula for $Z(t)$. The Riemann-Siegel formula [5], [6], [25] is an asymptotic expansion for $Z(t)$ (defined by (2.2)). The Riemann-Siegel formula is an improvement over the Euler-Maclaurin expansion for computing $Z(t)$ if $t$ is large, because the work required is $O(t^{1/4})$ instead of $O(t)$.

Let $\tau = t/(2\pi)$, $m = [\tau^{1/4}]$, and $z = 2(\tau^{1/2} - m) - 1$. Then the Riemann-Siegel formula with $n + 1$ terms in the asymptotic expansion is

$$Z(t) = \sum_{k=1}^{m} 2k^{-1/4} \cos \left[ t \ln(k) - \theta(t) \right]$$

$$+ (-1)^{m+1} \sum_{j=0}^{n} \Phi_j(z)(-1)^j \tau^{-j/2} + R_n(\tau),$$

where

$$R_n(\tau) = O(\tau^{-(2n+3)/4})$$

for $n \geq -1$ and $\tau > 0$. Here the $\Phi_j(z)$ are certain entire functions which may be expressed in terms of the derivatives of

$$\Phi_0(z) = \Phi(z) = \cos \left[ \pi(4z^2 + 3)/8 \right]/\cos(\pi z).$$

Expressions for $\Phi_1, \ldots, \Phi_{19}$ are given in the review of [5]. For our purposes it is sufficient to note that

$$\Phi_1(z) = \Phi(3)(z)/(12\pi^2)$$

and

$$\Phi_2(z) = \Phi(2)(z)/(16\pi^2) + \Phi(6)(z)/(288\pi^4).$$

To establish changes of sign of $Z(t)$ we need rigorous bounds on the error $R_n(\tau)$. Titchmarsh [27, p. 331] showed that

$$|R_0(\tau)| < \frac{3}{2} \tau^{-1/4} \text{ for } \tau > 125,$$

and Rosser et al. [23] used the bound

$$|R_2(\tau)| < 2.28\tau^{-7/4} \text{ for } \tau > 2000.*$$

This bound is extremely conservative; computation of $\max_{z \in [-1,1]} |\Phi_j(z)|$ for $j = 3, 4, \ldots$ (and computation of $R_2(\tau)$ for small $\tau$) indicates that the constants 2.28 and 2000 in (2.7) may be replaced by 0.006 and 10, respectively. In the computation described below we took $n = 2$ in (2.6) and used only the weak bound

$$|R_2(\tau)| < 3\tau^{-7/4} \text{ for } \tau > 2000.$$

*The number "2.88" appearing in [23] should have been "2.28".
The effect of rounding errors in accumulating the first sum in (2.6) was more of a problem than the inherent error (2.8); see Section 5.

3. Gram Blocks and the Littlewood-Turing Theorem. "Gram’s law" is the observation [7] that $Z(t)$ usually changes sign in each "Gram interval" $G_j = [g_j, g_{j+1}), j \geq -1$. A plausible explanation for this is that the leading ($k = 1$) term in (2.6) at $t = g_j$ is $2(-1)^j$. We call a Gram point $g_j$ good if $(-1)^j Z(g_j) > 0$, and bad otherwise. (The first bad Gram point is $g_{126}$.)

The concept of "Gram blocks" was introduced by Rosser et al. [23]. A “Gram block of length $k$" is an interval $B_j = [g_j, g_{j+k})$ such that $g_j$ and $g_{j+k}$ are good Gram points, $g_{j+1}, \ldots, g_{j+k-1}$ are bad Gram points, and $k \geq 1$. We say that $B_j$ satisfies “Rosser’s rule" if $Z(t)$ has at least $k$ zeros in $B_j$.

Rosser’s rule fails infinitely often [12], but it is still an extremely useful heuristic. The first exception is $B_{13,999,525}$ (see Table 3), so Lehman’s conjecture [12] that Rosser’s rule holds up to $g_{10,000,000}$ is correct.

Let $N(T)$ denote the number of zeros (counted according to their multiplicities) of $\xi(s)$ in the region $0 < I(s) \leq T$, and

$$S(t) = N(t) - 1 - \theta(t)/\pi.$$  

It is easy to show that Gram’s law holds in regions where $|S(t)| < 1$, and Rosser’s rule holds in regions where $|S(t)| < 2$. Thus, the success of these heuristics is closely related to the distribution of values of $S(t)$; see Lehman [12].

Turing [28] showed that the following theorem, based on an idea of Littlewood [15], could be used to bound $N(t)$ for certain values of $t$. We give Lehman’s version [12] of the theorem, as Turing’s constants $A$ and $B$ are larger than necessary, and his proof is incorrect.

**Theorem 3.1.** If $A = 0.114, B = 1.71, C = 168\pi$, and $C < u < v$, then

$$\left| \int_u^v S(t) \, dt \right| < A \ln(v) + B.$$

Since our program works with Gram blocks, the following consequence of Theorem 3.1 is extremely convenient.

**Theorem 3.2.** If $K$ consecutive Gram blocks with union $[g_n, g_p]$ satisfy Rosser’s rule, where

$$K \geq 0.0061 \left[ \ln (g_p) \right]^2 + 0.08 \ln (g_p),$$

then

$$N(g_n) < n + 1$$

and

$$N(g_p) > p + 1.$$  

**Proof.** If $g_n \leq 168\pi$ then (3.3) certainly holds [8], and (3.4) holds because Rosser’s rule is valid in $(g_{-1}, g_p)$. Thus, assume that $g_n > 168\pi$. 

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Since \( p-n \geq K \), it follows from (3.2) that

\[
(3.5) \quad K + \frac{1}{2}(p-n) > 0.152 \ln \left( \frac{g_p}{(2\pi)} \right) + 0.0091 \left[ \ln \left( \frac{g_p}{(2\pi)} \right) \right]^2.
\]

The result now follows from Theorem 4 of Lehman [12] (which is itself a consequence of Theorem 3.1).

4. The Computational Method. The first (and most expensive) part of the computational verification of \( H(n + 1) \) is the location of \( n + 1 \) sign changes of \( Z(t) \) in \((g_{-1}, g_n)\). Our program works in the following way. Suppose that \( j + 1 \) sign changes have been found in \((g_{-1}, g_j)\), where \( g_j \) is a good Gram point. Then \( Z(g_{j+1}), Z(g_{j+2}), \ldots \) are evaluated until the next good Gram point \( g_{j+k} \) is found. The program then evaluates \( Z(t) \) for various \( t \in (g_j, g_{j+k}) \), until either

(a) \( k \) sign changes are found in \( B_j \), when \( j \) is replaced by \( j + k \) and the process continues; or

(b) after a large number of evaluations of \( Z(t) \) the program gives up and calls for help.

Case (b) could arise because of a pair of very close zeros of \( Z(t) \) in \( B_j \) (or a multiple zero), or because \( B_j \) does not satisfy Rosser's rule. In fact, during the computation to \( n = 75,000,000 \), case (b) occurred only 15 times. In each case \( B_j \) contained \( k - 2 \) zeros of \( Z(t) \), and the preceding or following Gram block of length \( k' \) contained \( k' + 2 \) zeros of \( Z(t) \); see Table 3.

In this way we found the required \( n + 1 \) sign changes, establishing that \( N(g_n) \geq n + 1 \). By running the computation a little further we also showed that there are 4 Gram blocks in \([g_n, g_{n+5})\), and all of them satisfy Rosser's rule. Applying Theorem 3.2 gives \( N(g_n) \leq n + 1 \). Thus, \( N(g_n) = n + 1 \), and \( H(n + 1) \) holds. By locating the \( n \)th and \((n + 1)\)th zeros, it may be shown that \( N(32,585,736.4) = n = 75,000,000 \), as claimed in the abstract.

5. Computational Details. In Section 4 we glossed over an essential point: how can the sign of \( Z(t) \) be determined with certainty? If \( Z(t) \) is evaluated numerically from the Riemann-Siegel formula (2.6), the effect of rounding errors must be considered as well as the inherent error \( R_n(t) \).

5.1. Methods for Evaluating \( Z(t) \). It is desirable to have at least two methods for evaluating \( Z(t) \): a fast method which usually determines the sign of \( Z(t) \) unambiguously, and a slower but more accurate method which may be used if the fast method fails. We used the Euler-Maclaurin formula (2.5) both for small \( t \) and for checking purposes, but for brevity we shall only analyze the use of the Riemann-Siegel formula (2.6). We shall also assume that \( n = 2 \) in (2.6), and that \( t > 20,000\pi \). Our program uses the following two methods to evaluate the Riemann-Siegel sum

\[
(5.1) \quad s(t) = \sum_{k=1}^{m} 2k^{-\frac{1}{2}} \cos \left[ t \ln (k) - \theta(t) \right].
\]

Method A: The constants \( \ln (k) \), \( k = 1, 2, \ldots \), are precomputed (using double-precision) and stored in a table. For each value of \( k \), \( f = \frac{1}{2} \pi \) \( \frac{1}{2} \pi \) \([t \ln (k) - \theta(t)] \) is computed using double-precision, then truncated to single-precision. (Here \( \frac{1}{2} \pi \)
denotes the fractional part of $x$.) Then $\cos(2\pi f) = \cos \left[ t \ln(k) - \theta(t) \right]$ is approximated by a precomputed piecewise linear approximation, the result multiplied by the precomputed single-precision constant $2k^{-\frac{1}{2}}$, and the sum (5.1) accumulated in double precision.

**Method B:** The same as for Method A except that all computations are done using double-precision arithmetic, and $\cos(2\pi f)$ is evaluated as accurately as possible.

All computations were performed on a Univac 1100/42 computer, which has a 36-bit word and hardware single- and double-precision floating-point arithmetic (using 27- and 60-bit binary fractions, respectively).

5.2. **Rounding Error Analysis of Methods A and B.** The analysis is similar to that of Lehman [11] and Rosser et al. [23] so we shall omit detailed (and tedious) proofs of the following results. Recall that $m = \lceil r \rceil > 100$. Lemmas 5.1 and 5.2 are elementary, and Lemma 5.3 follows easily from them.

**Lemma 5.1.**

$$\sum_{k=1}^{m} k^{-\frac{1}{2}} \leq 2m^{\frac{1}{2}} \leq 2^n$$

and

$$\sum_{k=1}^{m} k^{-\frac{1}{2}} \ln(k) \leq 2m^{\frac{1}{2}} \ln(m) \leq m^{\frac{3}{2}} \ln(r).$$

**Lemma 5.2.**

$$\theta(t) < \pi \ln(r).$$

**Lemma 5.3.** Suppose that

$$|L(k) - \ln(k)| = \delta_1 \ln(k) \quad \text{for } k = 1, 2, \ldots, m,$$

$$|\tilde{\theta}(t) - \theta(t)| < \delta_2 \theta(t),$$

$$|\tilde{c}(x) - \cos(x)| = \delta_3 \quad \text{for } 0 \leq x < 2\pi,$$

and

$$\tilde{s}(t) = \sum_{k=1}^{m} 2k^{-\frac{1}{2}} \tilde{c}[t \cdot L(k) - \tilde{\theta}(t)].$$

Then

$$|\tilde{s}(t) - s(t)| \leq 4\pi r^{5/4} \ln(r)(\delta_1 + \delta_2) + 4rn^2 \delta_3.$$
For Method A we approximate \( \cos(2\pi x) \) for \( 0 < x < 1 \) using piecewise linear approximations on the intervals \([jh, (j + 1)h)\) for \( j = 0, \ldots, 1023 \) and \( h = 2^{-10} \). It is easy to show that, with exact arithmetic, the approximation error is bounded by \( 2^{-22}\pi^2 < 2.36 \times 10^{-6} \). Allowing for rounding errors in evaluating the linear approximations \( a + bx \) (with \( |a| < 3\pi/2, |b| < 2\pi, 0 < x < 1 \)) increases this bound slightly, giving \( \delta_3 < 2.6 \times 10^{-6} \).

For Method B it turns out that \( \delta_3 \) is negligible, because the errors in the cosine and logarithm evaluation are the same order of magnitude; but the error in the evaluation of \( \ln(k) \) contributes much more to the bound on the error in \( \Im(t) \) because it is amplified by the factor \( t > 20,000\pi \).

It is possible to allow for errors in evaluating \( \tau^{1/2} \) (and hence \( m \)) and the \( \Phi_j(z) \) in (2.6), but as these contribute little to the final error bound we shall omit the details. Collecting the results, and including the inherent error (2.8), we have the following bounds for the error in the computed value \( \hat{Z}(t) \) (rounded to single-precision) of \( Z(t) \):

\[
(5.2) |\hat{Z}(t) - Z(t)| \leq \begin{cases} 
(2 \times 10^{-5} + 5 \times 10^{-16}\tau \ln(t) + 3\tau^{-2})\tau^{1/4} & \text{for Method A,} \\
(5 \times 10^{-16}\tau \ln(t) + 3\tau^{-2})\tau^{1/4} + 8 \times 10^{-9}|\hat{Z}(t)| & \text{for Method B.}
\end{cases}
\]

These are the bounds actually used in the program, and are weaker than could be justified by the analysis sketched above.

5.3. Efficiency Considerations. When evaluating \( Z(t) \) our program always tries Method A first. If the computed \( |\hat{Z}(t)| \) is smaller than the bound (5.2), the sign of \( Z(t) \) cannot be guaranteed, so Method B is used. (Method B is also used once in 1000 evaluations to give a dynamic check on the consistency of the error bounds (5.2).) Occasionally Method B is unable to guarantee the sign of \( Z(t) \). If we are searching for sign changes in a Gram block and \( t \) is not a Gram point, we simply discard \( t \) and try another nearby point. If \( t \) is a Gram point the sign of \( Z(t) \) must be determined to ensure the accuracy of Tables 1–4 below. Thus, we occasionally use a multiple-precision arithmetic package [4] to evaluate \( Z(t) \) accurately at Gram points. (Actually, Method B always gives the correct sign of \( Z(g_n) \) for \( n \leq 75,000,000 \), even though the bound (5.2) is too weak to guarantee this.)

Nearly all the computation time is spent in the inner loop of Method A, so not much would be gained by speeding up Method B or increasing the accuracy of Method A. It also seems unlikely that the inner loop could be speeded up much without using a faster machine, as the loop compiles into only 19 machine instructions which execute in about 22 \( \mu \)sec. (The double-precision evaluation of \( \cos(2\pi x) \) using the standard library routine [1] takes about 79 \( \mu \)sec, and the inner loop of Method B takes about 150 \( \mu \)sec.)

To separate the first 75,000,000 zeros our program evaluated \( Z(t) \) at about 106,000,000 points. Thus, the heuristic of using Rosser’s rule is very efficient—the number of evaluations of \( Z(t) \) could not be reduced by more than 29 percent.

Our program requires about \( 35[n/\ln(n)]^{1/6} \) \( \mu \)sec of CPU time per Gram point near \( g_n, n \leq 10^8 \). Thus, the time required to verify \( H(n) \) is about \( 6.5 \times 10^{-9}n[n/\ln(n)]^{1/6} \).
hours. Our program is about 3.6 times faster than the CDC 3600 program of Rosser et al. [23], and about 11 times faster than the IBM 7090 program of Lehman [11]. This is roughly what one would expect, given the relative speeds of the different machines. (The times given for our program are approximate because of the variability of factors such as the ratio of primary to extended memory references, system load, etc.)

6. Summary of the Computational Results. During the course of the verification of \(H(75,000,001)\) we accumulated various statistics which are summarized in Tables 1 to 4. Table 1 gives the number \(J(k, n)\) of Gram blocks \(B_j = [g_j, g_{j+k})\) of length \(k \leq 7\) with \(0 \leq j < n\) and various \(n \leq 70,000,000\). (Note that \(B_{-1} = [g_{-1}, g_0)\) and the zero \(t_1 \in B_{-1}\) are excluded from the statistics given in Tables 1 to 4.) No Gram blocks of length greater than 7 were found.** The average block length up to \(n = 70,000,000\) is 1.1873, and increases slowly with \(n\). If the \(Z(g_j)\) had random independently distributed signs, then the average block length would be 2, so we conjecture that the average block length tends to a limit \(\lambda \leq 2\) as \(n \to \infty\).

In Table 2 we give the number of Gram intervals \(G_j = [g_j, g_{j+1})\), \(0 \leq j < n\), which contain exactly \(m\) zeros of \(Z(t)\), \(0 \leq m \leq 4\). About 74 percent of the Gram intervals up to \(n = 70,000,000\) contain precisely one zero, and this percentage decreases slowly with \(n\). We found only one Gram interval \((G_{61,331,768})\) which contains more than three zeros.

### Table 1

**Number of Gram blocks of given length**

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<th>(n)</th>
<th>(J(1,n))</th>
<th>(J(2,n))</th>
<th>(J(3,n))</th>
<th>(J(4,n))</th>
<th>(J(5,n))</th>
<th>(J(6,n))</th>
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<td>3,126,675</td>
<td>604,103</td>
<td>78,370</td>
<td>4,491</td>
<td>264</td>
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<tr>
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</tr>
<tr>
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<td>6,280,945</td>
<td>1,263,391</td>
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<td>668</td>
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<td>1,487,916</td>
<td>222,034</td>
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<td>849</td>
<td>30</td>
</tr>
</tbody>
</table>

**Blocks of length 8, e.g. \(B_{1,891,894,493}\), have been found by a different method (mentioned at the end of Section 6).**
Table 2

Number of Gram intervals containing exactly $m$ zeros

<table>
<thead>
<tr>
<th>$n$</th>
<th>$m=0$</th>
<th>$m=1$</th>
<th>$m=2$</th>
<th>$m=3$</th>
<th>$m=4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>100</td>
<td>3</td>
<td>3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>200</td>
<td>13</td>
<td>42</td>
<td>42</td>
<td></td>
<td></td>
</tr>
<tr>
<td>500</td>
<td>137</td>
<td>1,176</td>
<td>1,176</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>1,000</td>
<td>209</td>
<td>796</td>
<td>6</td>
<td>12</td>
<td></td>
</tr>
<tr>
<td>2,000</td>
<td>1,770</td>
<td>16,472</td>
<td>1,746</td>
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</tr>
<tr>
<td>5,000</td>
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<td>4,837</td>
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</tr>
<tr>
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<td>79,427</td>
<td>10,157</td>
<td>86</td>
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</tr>
<tr>
<td>20,000</td>
<td>21,528</td>
<td>157,153</td>
<td>21,110</td>
<td>209</td>
<td></td>
</tr>
<tr>
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<td>56,236</td>
<td>388,110</td>
<td>55,072</td>
<td>582</td>
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</tr>
<tr>
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<td>113,477</td>
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<tr>
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<tr>
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<td>3,778,577</td>
<td>600,087</td>
<td>7,083</td>
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<tr>
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<td>1,253,536</td>
<td>7,507,820</td>
<td>1,223,692</td>
<td>14,932</td>
<td></td>
</tr>
<tr>
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<td>2,550,785</td>
<td>14,929,745</td>
<td>2,488,155</td>
<td>31,315</td>
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<tr>
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<td>22,324,402</td>
<td>3,766,121</td>
<td>47,785</td>
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<td>5,052,747</td>
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<td>37,065,811</td>
<td>6,345,140</td>
<td>81,303</td>
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<tr>
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<td>44,418,273</td>
<td>7,643,577</td>
<td>98,191</td>
<td></td>
</tr>
<tr>
<td>100,000,000</td>
<td>9,174,803</td>
<td>51,765,709</td>
<td>8,944,174</td>
<td>115,313</td>
<td></td>
</tr>
</tbody>
</table>

Table 3

Exceptions to Rosser's rule

<table>
<thead>
<tr>
<th>$n$</th>
<th>Type</th>
<th>$S(t)$</th>
</tr>
</thead>
<tbody>
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<td>13,999,525</td>
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<td>-2.004138</td>
</tr>
<tr>
<td>30,783,329</td>
<td>1</td>
<td>-2.002594</td>
</tr>
<tr>
<td>30,930,927</td>
<td>2</td>
<td>+2.050625</td>
</tr>
<tr>
<td>37,592,215</td>
<td>1</td>
<td>-2.076426</td>
</tr>
<tr>
<td>40,870,156</td>
<td>1</td>
<td>-2.003797</td>
</tr>
<tr>
<td>43,628,107</td>
<td>1</td>
<td>-2.024243</td>
</tr>
<tr>
<td>46,082,042</td>
<td>1</td>
<td>-2.031132</td>
</tr>
<tr>
<td>46,875,667</td>
<td>1</td>
<td>-2.004600</td>
</tr>
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<td>49,624,541</td>
<td>2</td>
<td>+2.001841</td>
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<tr>
<td>50,799,238</td>
<td>1</td>
<td>-2.028778</td>
</tr>
<tr>
<td>55,221,454</td>
<td>2</td>
<td>+2.024210</td>
</tr>
<tr>
<td>56,948,780</td>
<td>2</td>
<td>+2.017714</td>
</tr>
<tr>
<td>60,515,663</td>
<td>1</td>
<td>-2.008143</td>
</tr>
<tr>
<td>61,331,766</td>
<td>3</td>
<td>-2.054298</td>
</tr>
<tr>
<td>69,784,844</td>
<td>2</td>
<td>+2.063683</td>
</tr>
</tbody>
</table>

**Type 1** is block $B_n$ of length 2 with no zeros, immediately followed by block $B_{n+2}$ of length 1 with 3 zeros.

**Type 2** is block $B_n$ of length 2 with no zeros, immediately preceded by block $B_{n-1}$ of length 1 with 3 zeros.

**Type 3** is block $B_n$ of length 2 with no zeros, immediately followed by block $B_{n+2}$ of length 2 with 4 zeros.

All exceptions to Rosser’s rule up to $B_{75,000,000}$ are included.
In Table 3 we list the 15 exceptions to Rosser's rule up to $B_{75,000,000}$. Each exception is associated with a small region where $|S(t)|$ exceeds 2, and the table gives the local extreme values of $S(t)$. Selberg [24] has shown that

$$S(t) = \Omega_{+} [(\ln t)^{1/3} (\ln \ln t)^{-7/3}],$$

and, assuming the Riemann hypothesis, Montgomery [19] has shown that

$$S(t) = \Omega_{+} [(\ln t)^{1/2} (\ln \ln t)^{-1/2}].$$

Probably

$$0 < \limsup_{t \to \infty} \frac{|S(t)|}{(\ln t)^{1/2}} < \infty;$$

see Lehman [12]. Unfortunately, it appears that the “interesting” region where $|S(t)|$ greatly exceeds 2 is well outside the range of feasible computation by the Riemann-Siegel formula, even by the method suggested at the end of this section.

Let $B_m = [g_m, g_{m+j}]$ be a Gram block which satisfies Rosser’s rule and has length $j \geq 2$. We say that $B_m$ is of type $(j, k)$ if $1 \leq k \leq j$ and $[g_{m+k-1}, g_{m+k}]$ contains at least two zeros of $Z(t)$. This is neither an unambiguous nor a complete classification, but it is sufficient to deal with all nontrivial Gram blocks up to $B_{75,000,000}$.
except for those noted in Table 3. The first occurrences of Gram blocks of various types are noted in Table 4. No blocks of type (7, 1) or (7, 7) occur up to $B_{1000000}$. Our program did not explicitly search for pairs of close zeros of $Z(t)$, but we did detect some such pairs when the program had difficulty in finding the expected number of sign changes in the Gram block containing them. For example,

$$t_{n+1} - t_n < 0.00053 \quad \text{and} \quad \max_{t \in (t_n, t_{n+1})} |Z(t)| < 0.00000248$$

for $n = 41,820,581$. This is a more extreme example of the phenomenon first observed by Lehmer [13], [14]. See also Montgomery [17], [18], [20], [21].

Our program regularly printed out the largest value of $|Z(g)|$ found so far. For example, $Z(g_{70,354,406}) > 79.6$, and the first 72 terms in the Riemann-Siegel sum (5.1) are positive at this point!

In all cases where an exception to Rosser’s rule was observed, there was a large local maximum of $|Z(t)|$ nearby. This suggests that “interesting” regions might be predicted by finding values of $t$ such that the first few terms in the Riemann-Siegel sum reinforce each other. Preliminary computations suggest that this is a promising approach. To verify the feasibility of such computations for Gram numbers near $10^{10}$ we ran our program (slightly modified) from $g_{n-500}$ to $g_{n+10000}$, where $n = 10^{10}$. All 8622 Gram blocks in this region satisfy Rosser’s rule and, using Theorem 3.2, we can show that $\rho_n, \rho_{n+1}, \ldots, \rho_{n+10000}$ are simple and lie on the critical line.

Acknowledgement. I wish to thank Enrico Bombieri, John Coates, Harold Edwards, R. Sherman Lehman, Derrick H. Lehmer, Hugh Montgomery, Lowell Schoenfeld and Daniel Shanks for many useful suggestions, and the Australian National University for the provision of computer time.

Added in Proof. By June 1, 1979 we had verified $H(81,000,001)$ and discovered three exceptions to Rosser’s rule in addition to those given in Table 3.

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Australian National University
Canberra, Australia


