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Very High Accuracy Chebyshev Expansions for the Basic Trigonometric Functions

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Abstract. Chebyshev expansion coefficients, accurate to forty decimal places, for the functions sine, cosine, and tangent, are tabulated. The methods used to generate the expansions are outlined and the ways in which accuracy of the tabulated coefficients were checked are noted.

1. Introduction. Sets of Chebyshev expansion coefficients for a number of functions have been available in the literature for some time. The most notable such published compilations are those due to Clenshaw [1] and Luke [2], both of which include expansions for the elementary trigonometric functions to be considered here. Most of these published sets of coefficients are for accuracies of up to twenty decimal places. Although this precision is adequate for most purposes, there are in existence a number of machines with floating-point systems which work to significantly higher accuracies; for instance double precision on ICL 1900 (22D) and CDC 6000 and 7000 (28D) and extended precision on IBM 370 and ICL 2900 (35D). Also, a number of micro-codable mini machines exist which for special applications may have floating-point arithmetics provided for accuracies higher than 20D. It was, therefore, considered desirable that some of the more basic expansions should be updated to provide for higher precisions. In this paper we present Chebyshev expansions for the basic trigonometric functions, sine(x), cosine(x) and tangent(x) for accuracies up to 40D.

In the next sections we present tables of the coefficients for the Chebyshev expansions for these functions, and the methods by which the expansions were generated and checked will be outlined. The results presented here arose out of the work being done by the author on the provision of transportable special function routines for the NAG library [3].

2. The Expansions. Five expansions will be tabulated, two each for sine and cosine and one for tangent.

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(i) $\mathrm{Cos}(\theta) = \Sigma'_{r=0}\, a_r T_r(t),^* \; -\pi/2 < \theta < \pi/2,\; t = 2(2\theta/\pi)^2 - 1$

```
Y=COS(X)  X=(0,PI/2)  T=2(2X/PI)*2 -1
MAX.ORD. 16      MAX.ACC.  40
   0          +9.440024315364695348953367757450019247285&-1
   1          -4.994032582704070874009136281196698573385&-1
   2           +2.79920796175476175122952951857706425403&-2
   3             -5.966951965488464992753518467530715589&-4
   4               +6.7043948699168401500864882299683098&-6
   5                 -4.65322958973195290109296441447595&-8
   6                  +2.193457658956733174654121081436&-10
   7                     -7.481648701033645762263723140&-13
   8                       +1.9322978458633275820681890&-15
   9                          -3.9101701216325903348848&-18
  10                             +6.3670401158338004758&-21
  11                                -8.5228860417326339&-24
  12                                  +9.5446630340576&-27
  13                                     -9.0744812452&-30
  14                                       +7.4159164&-33
  15                                          -5.2653&-36
  16                                            +3.3&-39
```

(ii) $\mathrm{Sin}(\theta) = \theta\; \Sigma'_{r=0}\, a_r T_r(t),\; -\pi/2 < \theta < \pi/2,\; t = 2(2\theta/\pi)^2 - 1$

```
Y=SIN(X)/X  X=(0,PI/2)  T=2(2X/PI)*2 -1
MAX.ORD. 16      MAX.ACC.  40
   0        +1.6250088450441268241600807200226765363200&+0
   1          -1.816031552372502018638303161580047542531&-1
   2            +5.8047092745986335594273417228579209601&-3
   3              -8.69543117793407571132123163531781403&-5
   4                +7.543701480888514810068399270295877&-7
   5                  -4.2671296650559611071268299059133&-9
   6                   +1.69804229454881681818247920130&-11
   7                      -5.01205788899618709295242855&-14
   8                        +1.141010266800106756282520&-16
   9                           -2.064375044247831339535&-19
  10                             +3.039695959187057766&-22
  11                                -3.713577341565809&-25
  12                                  +3.824861232465&-28
  13                           -[illegible].366226344&-31
  14                                       +2.560729&-34
  15                                          -1.701&-37
  16                                            +1.0&-40
```

*The notation $\Sigma'_{r=0}$ is used for the summation with the zeroth order coefficient divided by 2. The Chebyshev polynomials denoted by $T_r(t)$ are defined on the range $-1 < t < +1$ such that $T_r(t) = \mathrm{Cos}[r \arccos(t)]$. The polynomials defined on $0 < t < 1$ are denoted by $T^*_r(t)$, however this form has not been used except in passing. The two forms are related $T^*_r(t) = T_r(2t - 1)$. It should be noted that in each case the function actually expanded is an even function of θ and so as a polynomial in θ the expansions involve only even powers. In order to avoid using the zero odd order coefficients the expansion variable t has been used such that $t = 2(\theta)^2 - 1$. This mapping is suggested by the relations $T_{2r}(\theta) = T^*_r(\theta^2) = T_r(2\theta^2 - 1)$. It should also be noted that in the case of sine and tangent which are odd functions the zero at $\theta = 0$ is extracted explicitly by use of an auxiliary function of θ. This enables relative accuracy to be preserved for very small arguments.

(iii) $\mathrm{Cos}(\theta) = \Sigma'_{r=0}\, a_r\, T_r(t), -\pi/4 < \theta < \pi/4, t = 2(4\theta/\pi)^2 - 1$

```
 Y=COS(X)  X=(0,PI/4)  T=2(4X/PI)*2 -1
MAX.ORD. 13     MAX.ACC.  40
   0          +1.7032638274096160254008120301218521364006&+0
   1           -1.464366443908368633207963601399932496892&-1
   2             +1.9214493118146467969071454374507941650&-3
   3               -9.9649684898293000686691061842365839&-6
   4                 +2.75765956071873951864383935301798&-8
   5                   -4.73994980816484403744229510321&-11
   6                     +5.54954854148518274082726416&-14
   7                       -4.70970490651755595660385&-17
   8                         +3.02989760807937313389&-20
   9                           -1.52841493421461534&-23
  10                             +6.2074515435783&-27
  11                               -2.0733307230&-30
  12                                  +5.795385&-34
  13                                    -1.376&-37
```

(iv) $\mathrm{Sin}(\theta) = \theta\ \Sigma'_{r=0}\, a_r T_r(t), -\pi/4 < \theta < \pi/4, t = 2(4\theta/\pi)^2 - 1$

```
 Y=SIN(X)/X  X=(0,PI/4)  T=2(4X/PI)*2 -1
MAX.ORD. 13     MAX.ACC.  40
   0          +1.8995408831374895527365382264514878308603&+0
   1           -4.98404113370366640149298361896424447706&-2
   2             +3.877134361528273090286676638277634859&-4
   3               -1.4305800919320896335047551007573361&-6
   4                 +3.0736511554485672396773039335127&-9
   5                   -4.3183659742290589203243244892&-12
   6                     +4.2756499505778110669405188&-15
   7                       -3.1436071995800694144665&-18
   8                         +1.7839968296458613080&-21
   9                           -8.050514025743967&-25
  10                             +2.957818538845&-28
  11                               -9.01932035&-32
  12                                  +2.31921&-35
  13                                     -5.1&-39
```

These expansions may be used to obtain values for the functions for any argument by use of the following range reductions. In the case of expansions (i) and (ii) we may calculate $\mathrm{Cos}(x)$ or $\mathrm{Sin}(x)$ by letting $x = N\pi + \theta$, where N is an integer and $-\pi/2 < \theta < \pi/2$. That is $N = \mathrm{ROUND}(x/\pi)$** and $\theta = x - N\pi$.*** Thus,

$$\mathrm{Cos}(x) = \mathrm{Cos}(N\pi)\mathrm{Cos}(\theta) - \mathrm{Sin}(N\pi)\mathrm{Sin}(\theta) = (-1)^N \mathrm{Cos}(\theta),$$

$$\mathrm{Sin}(x) = \mathrm{Sin}(N\pi)\mathrm{Cos}(\theta) + \mathrm{Cos}(N\pi)\mathrm{Sin}(\theta) = (-1)^N \mathrm{Sin}(\theta).$$

For expansions (iii), (iv), and (v), use a similar technique but with a smaller final range.

** The operation ROUND(X) is used in the Algol 68 sense meaning the nearest integer to X (the method of breaking ties is not important but is usually round up).

*** These steps are critical in preserving accuracy for x outside the primary range ($N = 0$). See W. J. Cody and W. M. Waite, Argonne National Laboratory Report TM-321 (1977).

(v) $\tan(\theta) = \theta \; \Sigma'_{r=0} \, a_r T_r(t), \; -\pi/4 < \theta < \pi/4, \; t = 2(4\theta)/\pi)^2 - 1$

```
 Y=TAN(X)/X   X=(0,PI/4) T=32*X*X/PISQ-1
MAX.ORD. 35     MAX.ACC.  40
   0              +2.2538216985451410029559475108529445788219&+0
   1               +1.3592332341164079340020901741155361182299&-1
   2                +9.6589245192773693640800205793544317002&-3
   3                 +6.9285924900047896405225771090983834 94&-4
   4                  +4.9740846576329239906236827005446369 2&-5
   5                   +3.571202820568564301882240924017164 1&-6
   6                    +2.564006211463115523187713982408853&-7
   7                     +1.840873490919409256632901085067 81&-8
   8                      +1.321687691027024749212968059018 6&-9
   9                       +9.489290672876806280147425567 28&-11
  10                        +6.81300417238877665386106050 97&-12
  11                         +4.8915169169344160815217121 15&-13
  12                          +3.511951136883415388663438 80&-14
  13                           +2.5214674705846647241415768 8&-15
  14                            +1.810332193533590229497714&-16
  15                             +1.29976003623963450133820&-17
  16                               +9.33185720189928829001 5&-19
  17                                +6.69997202626594489584&-20
  18                                 +4.81036348730346244 70&-21
  19                                  +3.45368559589025341 9&-22
  20                                   +2.4796346942892340 4&-23
  21                                    +1.78029761146742 34&-24
  22                                     +1.2781961765158 76&-25
  23                                       +9.1770356548035&-27
  24                                        +6.588815156614&-28
  25                                         +4.7305564455 7&-29
  26                                          +3.396386717 9&-30
  27                                           +2.43849595 1&-31
  28                                            +1.75076132&-32
  29                                             +1.25699 01&-33
  30                                               +9.02478&-35
  31                                                +6.4795&-36
  32                                                 +4.652&-37
  33                                                  +3.34&-38
  34                                                   +2.4&-39
  35                                                   +2.0&-40
```

Let $x = N\pi/2 + \theta$, where N is an integer and $-\pi/4 < \theta < \pi/4$. That is, $N =$ ROUND$(2x/\pi)$, $\theta = x - N\pi/2$.† Then,

$$\mathrm{Cos}(x) = \mathrm{Cos}\left(N\frac{\pi}{2}\right)\mathrm{Cos}(\theta) - \mathrm{Sin}\left(N\frac{\pi}{2}\right)\mathrm{Sin}(\theta)$$

$$= \begin{Bmatrix} \mathrm{Cos}(\theta) & \text{as } N \text{ Modulo } 4 = 0 \\ -\mathrm{Sin}(\theta) & 1 \\ -\mathrm{Cos}(\theta) & 2 \\ \mathrm{Sin}(\theta) & 3 \end{Bmatrix},$$

$$\mathrm{Sin}(x) = \mathrm{Sin}\left(N\frac{\pi}{2}\right)\mathrm{Cos}(\theta) + \mathrm{Cos}\left(N\frac{\pi}{2}\right)\mathrm{Sin}(\theta)$$

$$= \begin{Bmatrix} \mathrm{Sin}\,\theta & \text{as } N \text{ modulo } 4 = 0 \\ \mathrm{Cos}\,\theta & 1 \\ -\mathrm{Sin}\,\theta & 2 \\ -\mathrm{Cos}\,\theta & 3 \end{Bmatrix},$$

† See footnote ***.

$$\tan(x) = \frac{\tan\left(\frac{N\pi}{2}\right) + \tan(\theta)}{1 - \tan\left(\frac{N\pi}{2}\right)\tan(\theta)}$$

$$= \begin{Bmatrix} \tan(\theta) & \text{as } N \text{ even} \\ -1/\tan(\theta) & \text{odd} \end{Bmatrix},$$

3. **Generation and Checking of Expansions.** Two methods were used to generate the four expansions for sine and cosine. The first and simplest of these "economizes" the power series, the second obtains a Chebyshev expansion solution to the appropriate differential equation.

If we have a function $y(u)$ which is represented to within our required accuracy by a truncated power series (note $-1 < u < +1$), then the resulting simple polynomial can be simply rearranged expressing it in terms of the Chebyshev polynomials. That is, we say

$$y(u) = \sum_{r=0}^{N} b_r u^r \equiv {\sum_{s=0}^{N}}' a_s T_s(u). \tag{1}$$

It is relatively easy to show that the Chebyshev coefficients a_s can be calculated from the power series coefficients as

$$a_s = \sum_{r=s}^{N} b_r C_{rs}, \tag{2}$$

where the factors C_{rs} can be generated recursively from the relation

$$C_{rs} = ½[C_{r-1,s+1} + C_{r-1,|s-1|}], \qquad C_{00} = 2. \tag{3}$$

This recursion and summation is extremely straightforward to program and provided arithmetic working and truncation accuracies for the power series are chosen to allow a reasonable margin for error (a few decimal places over the required accuracy)†† the resulting a_s provide accurate estimates of the Chebyshev expansion coefficients. As is well known, the Chebyshev expansion coefficients will normally be much smaller for large order than the corresponding power series coefficients ($a_s \sim b_s/2^{s-1}$ or better). Therefore, in order to represent the required function the number of terms that need to be retained in the Chebyshev form is usually very much smaller than for the original power series. Turning to the actual functions in question, the actual expansion variables used were of the form $t = 2(\theta/\lambda)^2 - 1$ where θ was allowed to range from $-\lambda$ to $+\lambda$. This expansion form is used because all the functions actually expanded are even functions of θ and, hence, involve only even powers or orders of polynomial. It should be noted that

$$\sum_{r=0}' a_r T_r(t) = \sum_{r=0}' a_r T_r^*(\theta^2/\lambda^2) = \sum_{r=0}' a_r T_{2r}(\theta/\lambda). \tag{4}$$

††The conversion of a simple polynomial to Chebyshev form is a numerically stable process so being ultra cautious and working to accuracies and tolerances a few decimal places better than actually required should achieve results of sufficient accuracy.

Thus taking the expansion variable u as θ/λ we would obtain for $\mathrm{Cos}(\theta)$, say

$$b_r = \begin{cases} (-1)^{r/2}(\lambda/r!)^r & r \text{ even} \\ 0 & r \text{ odd} \end{cases} . \tag{5}$$

This will generate a set of Chebyshev coefficients a_s in which the odd orders are zero and the even, $s = 2r$, may be equated with the required a_r in (4). This process can be performed for any value of λ. The values used, for obvious reasons were $\pi/2$ and $\pi/4$.

The whole process was programmed in Algol 68 using the facilities for multiple length arithmetic provided by Mlaritha [4]. The working accuracy used was 48 decimal figures and the power series was truncated at $N = 50$ with a truncation error of order 10^{-55} for the cosine series $\lambda = \pi/2$. Similar accuracy criteria were used for the other series which follow similar patterns. The resulting expansion coefficients were rounded to forty decimal places and written to permanent file storage.

The solution of differential equation technique was outlined in [5]. It is easy to show that for the cosine and sine the functions actually being expanded satisfy the following linear differential equations in terms of the expansion variable $t = 2(\theta/\lambda)^2 - 1$.

For cosine

$$2(1 + t)\ddot{y} + \dot{y} + \frac{\lambda^2}{4} y = 0, \qquad y(-1) = 1, \dot{y}(-1) = \lambda^2/4. \tag{6}$$

For sine

$$2(1 + t)\ddot{y} + 3\dot{y} + \frac{\lambda^2}{4} y = 0, \qquad y(-1) = 1, \dot{y}(-1) = -\lambda^2/12. \tag{7}$$

These differential equations can be used to find the Chebyshev expansion coefficients for the solution $y(t)$ in each case. The basic method is essentially that outlined in Fox and Parker [6]. Assuming that $y(t)$ can be represented by a Chebyshev expansion, then from the differential equation it can be shown that the coefficient a_r satisfies an infinite set of linear equations, $A\mathbf{a} = \mathbf{b}$. The first two rows of the matrix A and vector $\mathbf{b}$ arise from the boundary conditions and the differential equation determines the subsequent rows. This set of equations may be solved approximately by assuming that all coefficients a_r with r greater than some N are negligible and, hence, may be approximated by zero. This reduces the infinite set of equations to an easily solved finite set. In fact, the required value of N can be determined as part of the solution process. The matrix A is normally diagonally dominant, except for the first few rows and a simple Crout LU decomposition technique will provide a sufficiently accurate solution.

The LU decomposition is independent of any increase in order of the system, and so the process can be performed iteratively. At any order N the next partial row and column can be formed, and the decomposition extended from order $N - 1$ to N. The forward substitution can also be similarly extended. The first step of the backward substitution, which is trivial, then gives an estimate of the value of the coefficient a_N included in the nonzero set. The process can be terminated when a_N is sufficiently small; in fact, the process is normally terminated when two successive coefficients are less than the required tolerance. At this point the full solution set is generated by backward substitution.

This differential equation solving technique has been programmed as a general package called CHEBEXP [7], using Algol 68 and the multiple precision arithmetic capability provided by Mlaritha [4]. The differential systems (6) and (7) were solved using a working accuracy of 48 decimal figures and a termination tolerance of 10^{-45}. The resulting coefficients were rounded to 40D and stored in permanent file storage.

The use of these two independent methods for obtaining the same expansion coefficients gives an excellent check on the accuracy of the final expansions. The expansions tabulated in the previous section were those actually produced by the first technique but in all cases the results from the differential equation method were identical. As a further check the Chebyshev expansions were summed for selected values of t where the sum is known exactly, or at least to accuracies greater than 40D. In all cases the resulting sums were correct to within a few parts in the fortieth decimal place, the error being entirely due to the effects of rounding to 40D [8].

Basically neither of the above techniques is practical for the $\tan(x)$ expansions. The expansion function $\tan(\theta)/\theta$ does not satisfy a simple linear differential equation and hence the second of the two techniques is not possible. The power series of $\tan(\theta)/\theta$ has coefficients which involve the Bernoulli numbers; and hence, this does not provide a practical method owing to difficulties in obtaining accurate values for the high order Bernoulli numbers involved. The method actually employed made use of the fact that we already had high accuracy values for the Chebyshev expansion coefficients for $\mathrm{Sin}(\theta)/\theta$ and $\mathrm{Cos}(\theta)$ for the same expansion range and variable.

If

(8) $$y(t) = \sum_{r=0}{}' a_r T_r(t) = \tan(\theta)/\theta = \frac{\mathrm{Sin}(\theta)/\theta}{\mathrm{Cos}(\theta)},$$

then we can write

(9) $$\mathrm{Cos}(\theta)\, y(t) = \mathrm{Sin}\ \theta/\theta,$$

that is,

(10) $$\sum_{r=0}{}' C_r T_r(t) y(t) = \sum_{r=0}{}' S_r T_r(t),$$

where C_r and S_r are the Chebyshev expansion coefficients of $\mathrm{Cos}(\theta)$ and $\mathrm{Sin}(\theta)/\theta$, respectively. It is again easy to show that this implies that the a_r satisfy an infinite set of linear equations. This set of equations can then be "solved" by techniques similar to those outlined above. This was in fact done. The coefficients C_r and S_r were taken from runs of the previous techniques which were designed to produce coefficients accurate to about 50D. The tangent expansion coefficients were then generated using a working accuracy of 48D and termination tolerance of 10^{-45}; and as before, the resulting coefficients were rounded to 40D and stored. In this case we did not have the luxury of a second method for the expansion; but again, check sums were calculated at selected points, and the sums were found to be correct to the expected accuracy.

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†††This book gives FORTRAN programs which use an efficient backward recursion technique for expanding various hypergeometric functions as Chebyshev series. These techniques could be used for Sin and Cos and probably also with some modification for tan. However, the FORTRAN programs can only be used to give coefficients accurate to something less than the greatest working precision available. (REAL *16 on IBM or ICL 2900.)