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On Maximal Finite Irreducible Subgroups of $GL(n, \mathbf{Z})$ III. The Nine Dimensional Case

By Wilhelm Plesken and Michael Pohst

Abstract. All maximal finite absolutely irreducible subgroups of $GL(9, \mathbf{Z})$ are determined up to conjugacy in $GL(9, \mathbf{Z})$.

1. Introduction. We determine all maximal finite irreducible subgroups of $GL(9, \mathbf{Z})$ up to $\mathbf{Z}$-equivalence. (Here and in the following, irreducible means $\mathbf{C}$-irreducible.)* There are 20 $\mathbf{Z}$-classes, a set of representatives of which is described in Section 4, Theorem (4.1). The quadratic forms fixed by these groups are listed in Section 3.

We employ the methods developed in Part I [7]. In Section 2 the minimal irreducible finite subgroups of $GL(9, \mathbf{Z})$ are determined up to $\mathbf{Q}$-equivalence; there are only three classes. The $\mathbf{Z}$-classes of the natural representations of the three groups, respectively the $\prec$-maximal centerings of the corresponding lattices, were electronically computed on the CDC Cyber 76 at the Rechenzentrum der Universität zu Köln. They are listed in Section 3.

In Part V of these series of papers we shall present a full set of representatives of the $\mathbf{Z}$-classes of the maximal finite irreducible subgroups of $GL(n, \mathbf{Z})$ for $n \leqslant 9$ by listing generators of the groups, the corresponding quadratic forms fixed by these groups, and the shortest vectors of these forms.

2. The Minimal Irreducible Finite Subgroups of $GL(9, \mathbf{Z})$. The minimal irreducible finite subgroups of $GL(9, \mathbf{Z})$ which are solvable will turn out to be rationally equivalent to a monomial group. Therefore (see Theorem (3.2) in Part I [7]) we need the minimal transitive permutation groups of degree 9. These are conjugate in S_9 to

$$P_1 := \langle\langle(123456789)\rangle\rangle \cong C_9 \quad \text{and}$$

$$P_2 := \langle\langle(123)(456)(789), (147)(258)(369)\rangle\rangle \cong C_3 \times C_3$$

as one easily sees from the following lemma.

(2.1) LEMMA. *A minimal transitive permutation group P of prime power degree p^k is of prime power order.*

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*For $n = 9$ $\mathbf{Q}$- and $\mathbf{C}$-irreducibility of finite subgroups of $GL(n, \mathbf{Z})$ coincide, as can be seen by standard arguments using the results in [1].

Proof. Let S be a p-Sylow-subgroup of P and P_0 the stabilizer of an element of the set Ω on which P acts. We must only show that S also acts transitively on Ω which is tantamount to $SP_0 = P$. By counting cosets one obtains

$$|SP_0| = \frac{|S||P_0|}{|S \cap P_0|}.$$

Since $|S \cap P_0|$ divides $|S|$ as well as $|P_0|$, we get $|SP_0| = |P|$; hence $SP_0 = P$. Q.E.D.

The methods employed in Parts I and II [7], [8] now yield:

(2.2) LEMMA. *The solvable minimal irreducible finite subgroups of $GL(9, \mathbf{Z})$ are rationally equivalent to*

$$G_1 = \langle \mathrm{diag}(-1, 1, 1, -1, 1, 1, 1, 1, 1), D(P_1)\rangle, \quad |G_1| = 2^6 9 \quad \textit{or to}$$

$$G_2 = \langle \mathrm{diag}(-1, 1, 1, -1, -1, 1, 1, -1, 1), D(P_2)\rangle, \quad |G_2| = 2^4 9.$$

Here and in the following D denotes the natural permutation representation of S_9, $D\colon S_9 \longrightarrow GL(9, \mathbf{Z})$ with $D(\pi)e_i = e_{\pi(i)}$ for $\pi \in S_9$, $i = 1, \ldots, 9$, $e_1, \ldots, e_9$ the standard basis of $\mathbf{Z}^{9 \times 1}$.

Proof. Let G be a minimal irreducible subgroup of $GL(9, \mathbf{Z})$, Δ the natural representation of G, and N a maximal abelian normal subgroup of G. As in Part II [8] we apply Theorem (3.1) of Part I [7]. Thus, we may assume that the restriction $\Delta|_N$ is equal to $\Gamma_1 \dotplus \cdots \dotplus \Gamma_r$, where $\Gamma_1, \ldots, \Gamma_r$ are integral representations of N satisfying $\Gamma_i \sim_{\mathbf{Q}} k\Delta_i$ ($i = 1, \ldots, r$; $k \in \mathbf{N}$) and $\Gamma_1(N) = \cdots = \Gamma_r(N)$. The Δ_i ($i = 1, \ldots, r$) are inequivalent $\mathbf{Q}$-irreducible integral representations of N all of the same degree m.

Since the $\mathbf{Q}$-enveloping algebra of $\Delta_1(N)$ is a cyclotomic field, m has to be even or equal to 1, hence $m = 1$ because of $m|9$. There remain three possible solutions of the degree equation $9 = krm$, namely (i) $k = m = 1$, $r = 9$, (ii) $k = r = 3$, $m = 1$, (iii) $m = r = 1$, $k = 9$.

Case (i). According to Theorem (3.2) in Part I [7] and Lemma (2.1) G is a subgroup of the group generated by all diagonal matrices and $D(P_1)$ or by all diagonal matrices and $D(P_2)$. Moreover, by the Schur-Zassenhaus Theorem G must split over N. Again by Theorem (3.2) in Part I, N is minimal with the properties (α) $N \subseteq \{\mathrm{diag}(a_1, \ldots, a_9) | a_i = \pm 1, (i = 1, \ldots, 9)\}$ and N is invariant under conjugation by $D(P_1)$ or $D(P_2)$, respectively; (β) the projections $N \longrightarrow \{\pm 1\}$: $\mathrm{diag}(a_1, \ldots, a_9) \longrightarrow a_i$ ($i = 1, \ldots, 9$) are pairwise unequal. Then G can be chosen as $\langle N, D(P_1)\rangle$ or $\langle N, D(P_2)\rangle$, respectively. For $D(P_1) \subseteq G$ one easily sees that there is only one possibility for N, whereas for $D(P_2) \subseteq G$ there are six possible candidates for N. These, however, are all conjugate under the normalizer of $D(P_2)$ in $GL(9, \mathbf{Z})$. Hence, we end up with two groups in Case (i).

Case (ii). Here N must be a subgroup of index 1 or 2 of $\{\mathrm{diag}(\pm I_3, \pm I_3, \pm I_3)\}$. Therefore, the nonzero quadratic forms which are fixed by G have matrices $\mathrm{diag}(A_1, A_2, A_3)$ with symmetric definite $A_i \in \mathbf{Z}^{3 \times 3}$ ($i = 1, 2, 3$). G can be chosen in such a way that $A_1 = A_2 = A_3$ and that G is a subgroup of $H \sim S_3$, where H is the automorphism group of A_1 [2]. Since G is minimal irreducible of odd degree, it can-

not have a normal subgroup of index 2, hence $G \leqslant H \sim C_3$. But all finite irreducible subgroups of $GL(3, \mathbf{Z})$ are isomorphic to subgroups of $C_2 \times S_4$. Thus, N cannot be a maximal abelian normal subgroup of G; and there is no group in Case (ii).

Case (iii). Either N must be trivial and G, therefore, nonsolvable or $N = \langle -I_9 \rangle$; and G has a normal subgroup $G \cap SL(9, \mathbf{Z})$ of index 2 and, hence, has a proper irreducible subgroup. Q.E.D.

For the determination of the nonsolvable groups G we use the classification of the primitive finite subgroups of $SL(9, \mathbf{C})$ by Feit [3] and Huffman and Wales [4].

(2.3) LEMMA. *The nonsolvable minimal irreducible finite subgroups of $GL(9, \mathbf{Z})$ are rationally equivalent to*

$$G_3 = \langle\langle (e_4, e_8, e_6, e_5, e_7, e_9, e_0, e_3, e_2), (e_0, e_9, e_1, e_8, e_7, e_2, e_6, e_3, e_5) \rangle\rangle$$
$$\cong A_6 \cong PSL(2, 9).$$

As in Lemma (2.2) $e_1, \ldots, e_9$ *denote the standard basis of* $\mathbf{Z}^{9 \times 1}$, $e_0 = -\Sigma_{i=1}^{9} e_i$.

Proof. We use the terminology of the proof of Lemma (2.2). Clearly, only Case (iii) can occur. Since G must be contained in $SL(9, \mathbf{Z})$, we have $N = \langle 1 \rangle$.

We assume that G has a minimal normal subgroup $M \neq \langle 1 \rangle$. Then M is characteristically simple, and $\Delta|_N$ becomes (C-)reducible with 3-dimensional (C-)constituents. A comparison with Blichfeldt's list of primitive subgroups of $SL(3, \mathbf{C})$ [1] shows that this is impossible because of rationality conditions for the characters. Therefore, G must be simple and, hence, quasi-primitive. By [3] and [4] G has to be isomorphic to A_6 or A_{10}. But the 9-dimensional irreducible representations of A_6 and A_{10} are both obtained by reducing their permutation representations of degree 10 (note that $A_6 \cong PSL(2, 9)$). So $G \cong A_{10}$ is not minimal irreducible. Since $PSL(2, 9)$ does not have any doubly transitive subgroups acting on 10 points, A_6 yields a minimal irreducible subgroup G_3 of $GL(9, \mathbf{Z})$. Q.E.D.

3. Computation of the Z-Classes in the 9-Dimensional Case. The centerings of the minimal irreducible subgroups G_1, G_2, G_3 of $GL(9, \mathbf{Z})$ yield 20 quadratic forms F, whose automorphism groups are the maximal finite irreducible subgroups of $GL(9, \mathbf{Z})$. Let J_n denote the $n \times n$ matrix with all entries equal to 1. Then the forms are given by the following matrices F_i, their determinants by d_i ($i = 1, \ldots, 20$):

$$F_1 = I_9, \quad d_1 = 1; \quad F_2 = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 \end{pmatrix}, \quad d_2 = 4;$$

$$F_3 = \begin{pmatrix} 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 9 \end{pmatrix}, \quad d_3 = 4^8;$$

$$F_4 = I_3 \otimes (I_3 + J_3), \quad d_4 = 4^3;$$

$$F_5 = I_3 \otimes (4I_3 - J_3), \quad d_5 = 4^6;$$

$$F_6 = \begin{pmatrix} 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 3 \end{pmatrix}, \quad d_6 = 4^2;$$

$$F_7 = \begin{pmatrix} 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 2 & 2 \\ 0 & 0 & 0 & 2 & 2 & 2 & 2 & 6 & 1 \\ 2 & 2 & 2 & 0 & 0 & 2 & 2 & 1 & 6 \end{pmatrix}, \quad d_7 = 4^7;$$

$$F_8 = \begin{pmatrix} 3 & -1 & -1 & 0 & 0 & 0 & -1 & 1 & 1 \\ -1 & 3 & -1 & 0 & 0 & 0 & 1 & -1 & 1 \\ -1 & -1 & 3 & 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 3 & -1 & -1 & 1 & 1 & -1 \\ 0 & 0 & 0 & -1 & 3 & -1 & -1 & 1 & 1 \\ 0 & 0 & 0 & -1 & -1 & 3 & 1 & -1 & 1 \\ -1 & 1 & 1 & 1 & -1 & 1 & 3 & 0 & 0 \\ 1 & -1 & 1 & 1 & 1 & -1 & 0 & 3 & 0 \\ 1 & 1 & -1 & -1 & 1 & 1 & 0 & 0 & 3 \end{pmatrix}, \quad d_8 = 4^4;$$

$$F_9 = \begin{pmatrix} 4 & 2 & 2 & 2 & 1 & 1 & 2 & 1 & 1 \\ 2 & 4 & 2 & 1 & 2 & 1 & 1 & 2 & 1 \\ 2 & 2 & 4 & 1 & 1 & 2 & 1 & 1 & 2 \\ 2 & 1 & 1 & 4 & 2 & 2 & 2 & 1 & -1 \\ 1 & 2 & 1 & 2 & 4 & 2 & 1 & 2 & 1 \\ 1 & 1 & 2 & 2 & 2 & 4 & 1 & 1 & 0 \\ 2 & 1 & 1 & 2 & 1 & 1 & 4 & 2 & 0 \\ 1 & 2 & 1 & 1 & 2 & 1 & 2 & 4 & 2 \\ 1 & 1 & 2 & -1 & 1 & 0 & 0 & 2 & 4 \end{pmatrix}, \quad d_9 = 4^5;$$

$$F_{10} = (I_3 + J_3) \otimes (I_3 + J_3), \quad d_{10} = 4^4 16;$$

$$F_{11} = (4I_3 - J_3) \otimes (4I_3 - J_3), \quad d_{11} = 4^4 16^4;$$

$$F_{12} = \begin{pmatrix} 6 & -2 & -2 & 3 & -1 & -1 & 3 & -1 & -1 \\ -2 & 6 & -2 & -1 & 3 & -1 & -1 & 3 & -1 \\ -2 & -2 & 6 & -1 & -1 & 3 & -1 & -1 & 3 \\ 3 & -1 & -1 & 6 & -2 & -2 & 1 & 1 & 1 \\ -1 & 3 & -1 & -2 & 6 & -2 & 1 & 1 & 1 \\ -1 & -1 & 3 & -2 & -2 & 6 & 1 & 1 & 1 \\ 3 & -1 & -1 & 1 & 1 & 1 & 6 & 2 & 2 \\ -1 & 3 & -1 & 1 & 1 & 1 & 2 & 6 & 2 \\ -1 & -1 & 3 & 1 & 1 & 1 & 2 & 2 & 6 \end{pmatrix}, \quad d_{12} = 4^6 16;$$

$$F_{13} = \begin{pmatrix} 8 & 4 & 4 & 0 & 0 & 0 & 0 & 0 & 4 \\ 4 & 8 & 4 & 0 & 0 & 0 & 0 & 0 & 4 \\ 4 & 4 & 8 & 0 & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 8 & 4 & 4 & 0 & 0 & 4 \\ 0 & 0 & 0 & 4 & 8 & 4 & 0 & 0 & 4 \\ 0 & 0 & 0 & 4 & 4 & 8 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 8 & 4 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 8 & 4 \\ 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 9 \end{pmatrix}, \quad d_{13} = 4^6 16^2;$$

$$F_{14} = (I_3 + J_3) \otimes (4I_3 - J_3), \quad d_{14} = 4^5 16^2;$$

$$F_{15} = 10I_9 - J_9, \quad d_{15} = 10^8;$$

$$F_{16} = I_9 + J_9, \quad d_{16} = 10;$$

$$F_{17} = \begin{pmatrix} 8 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & -3 \\ 3 & 8 & 3 & 3 & 3 & 3 & 3 & 3 & -3 \\ 3 & 3 & 8 & 3 & 3 & 3 & 3 & 3 & 2 \\ 3 & 3 & 3 & 8 & 3 & 3 & 3 & 3 & 2 \\ 3 & 3 & 3 & 3 & 8 & 3 & 3 & 3 & 2 \\ 3 & 3 & 3 & 3 & 3 & 8 & 3 & 3 & 2 \\ 3 & 3 & 3 & 3 & 3 & 3 & 8 & 3 & 2 \\ 3 & 3 & 3 & 3 & 3 & 3 & 3 & 8 & 2 \\ -3 & -3 & 2 & 2 & 2 & 2 & 2 & 2 & 8 \end{pmatrix}, \quad d_{17} = 5^7 10;$$

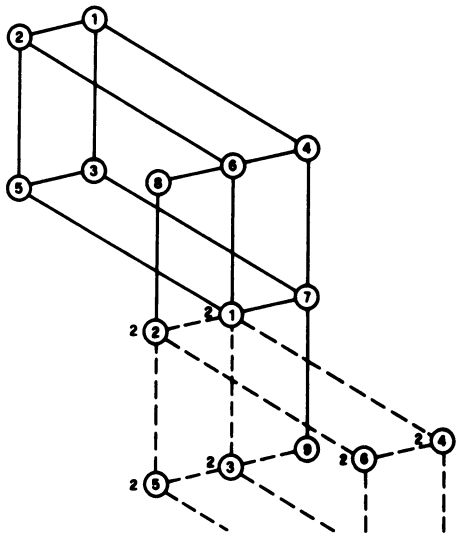
$$F_{18} = \begin{pmatrix} 4 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 0 \\ 2 & 4 & 2 & 2 & 2 & 2 & 2 & 2 & 0 \\ 2 & 2 & 4 & 2 & 2 & 2 & 2 & 2 & 0 \\ 2 & 2 & 2 & 4 & 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 4 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 & 4 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 4 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 4 & 2 \\ 0 & 0 & 0 & 2 & 2 & 2 & 2 & 2 & 5 \end{pmatrix}, \quad d_{18} = 2^7 10;$$

$$F_{19} = \begin{pmatrix} 12 & 2 & 2 & -3 & 3 & 3 & -3 & -2 & 3 \\ 2 & 12 & -3 & 2 & 3 & 3 & -3 & 3 & -2 \\ 2 & -3 & 12 & 2 & -2 & 3 & 2 & 3 & -2 \\ -3 & 2 & 2 & 12 & 3 & -2 & 2 & 3 & 3 \\ 3 & 3 & -2 & 3 & 12 & 2 & 3 & 2 & 2 \\ 3 & 3 & 3 & -2 & 2 & 12 & 3 & -3 & -3 \\ -3 & -3 & 2 & 2 & 3 & 3 & 12 & -2 & 3 \\ -2 & 3 & 3 & 3 & 2 & -3 & -2 & 12 & -3 \\ 3 & -2 & -2 & 3 & 2 & -3 & 3 & -3 & 12 \end{pmatrix}, \quad d_{19} = 5^3 10 \cdot 20^4;$$

$$F_{20} = \begin{pmatrix} 4 & -2 & -1 & -1 & 0 & -1 & 1 & 2 & -1 \\ -2 & 4 & 2 & 2 & 0 & 0 & 1 & -1 & 2 \\ -1 & 2 & 4 & 0 & -1 & 0 & 0 & 1 & 1 \\ -1 & 2 & 0 & 4 & -1 & 0 & 2 & -1 & 1 \\ 0 & 0 & -1 & -1 & 4 & 1 & 0 & -1 & -1 \\ -1 & 0 & 0 & 0 & 1 & 4 & 0 & 1 & -1 \\ 1 & 1 & 0 & 2 & 0 & 0 & 4 & 1 & 2 \\ 2 & -1 & 1 & -1 & -1 & 1 & 1 & 4 & 0 \\ -1 & 2 & 1 & 1 & -1 & -1 & 2 & 0 & 4 \end{pmatrix}, \quad d = 2 \cdot 4^3 \cdot 20.$$

Now we describe the $\prec$-maximal centerings of the minimal irreducible groups G_1, G_2, G_3 which were computed by machine. If L is an irreducible $\mathbf{Z}G$-representation module and H a $\prec$-maximal centering of L, then $M^{\#}$ denotes the unique $\prec$-maximal centering of L which belongs to the inverse transposed representation coming from M [7].

Lattice of centerings for G_1:



$L_1^{\#} = L_1$, $L_2^{\#} = L_7$, $L_3^{\#} = L_6$,
$L_4^{\#} = L_5$, $L_8^{\#} = L_9$.

(Read $\textcircled{i} = L_i$ $(i = 1, \ldots, 9)$.)

Bases of the centering (numbers in brackets refer to the corresponding quadratic form F_i):

(1) $B(L_1) = I_8$,

(2) $B(L_2) = (x_1, h_1x_1, \ldots, h_1^7x_1, y_1)$ with $x_1^T = (1, 1, 0, \ldots, 0)$, $y_1^T = (0, \ldots, 0, 1, -1)$ and $h_1 = D((123456789))$,

$$B(L_3) = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{6}$$

(5) $B(L_4) = D((24)(37)(68))\,(I_3 \otimes (J_3 - 2I_3))$,

(4) $B(L_5) = D((24)(37)(68))\,(I_3 \otimes (J_3 - I_3))$,

(7) $B(L_6) = (2e_1, 2e_4, 2e_7, 2e_3, 2e_6, 2e_2, 2e_5, x_2, y_2)$ with $x_2^T = (0, 1, 1, 0, 1, 1, 0, 1, 1)$, $y_2^T = (1, 1, 0, 1, 1, 0, 1, -1, 0)$ (the e_i are defined in (2.3)),

(3) $B(L_7) = (2e_1, \ldots, 2e_8, -e_0)$,

$$(12) \qquad B(L_8) = \begin{pmatrix} 1 & -1 & -1 & 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & -1 & -1 & -1 & -1 & -1 \\ -1 & 1 & 1 & -1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 1 & -1 & 0 & 0 & 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & -1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 1 & -1 & 1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & -1 & -1 & 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & 1 & 1 & -1 & 0 & 0 & 0 \end{pmatrix},$$

(13) $B(L_9) = (2(e_1 + e_4), 2(e_1 + e_7), 2(e_4 + e_7), 2(e_2 + e_5), 2(e_2 + e_8), 2(e_5 + e_8), 2(e_3 + e_6), 2(e_3 + e_9), -e_0)$.

We proceed to the $\prec$-maximal centerings of G_2: $M_1^\# = M_1$, $M_2^\# = M_{32}$, $M_3^\# = M_{31}$, $M_4^\# = M_{30}$, $M_5^\# = M_{29}$, $M_6^\# = M_{28}$, $M_7^\# = M_{26}$, $M_8^\# = M_{25}$, $M_9^\# = M_{24}$, $M_{10}^\# = M_{23}$, $M_{11}^\# = M_{22}$, $M_{12}^\# = M_{21}$, $M_{13}^\# = M_{20}$, $M_{14}^\# = M_{19}$, $M_{15}^\# = M_{18}$, $M_{16}^\# = M_{17}$, $M_{27}^\# = M_{39}$, $M_{33}^\# = M_{38}$, $M_{34}^\# = M_{37}$, $M_{35}^\# = M_{36}$. (See figure on next page.)

Because of the large number of centerings we do not give a list of all bases but describe how to obtain them.

(1) $B(M_1) = I_9$,

(2) $B(M_2) = B(L_2)$,

(6) $B(M_3) = D((2734)(5896))B(L_3)$, $B(M_i)$ $(i = 4, 5, 6)$ are given by $D(\pi)B(M_3)$, where π is a suitable element of the normalizer of P_2 in S_9,

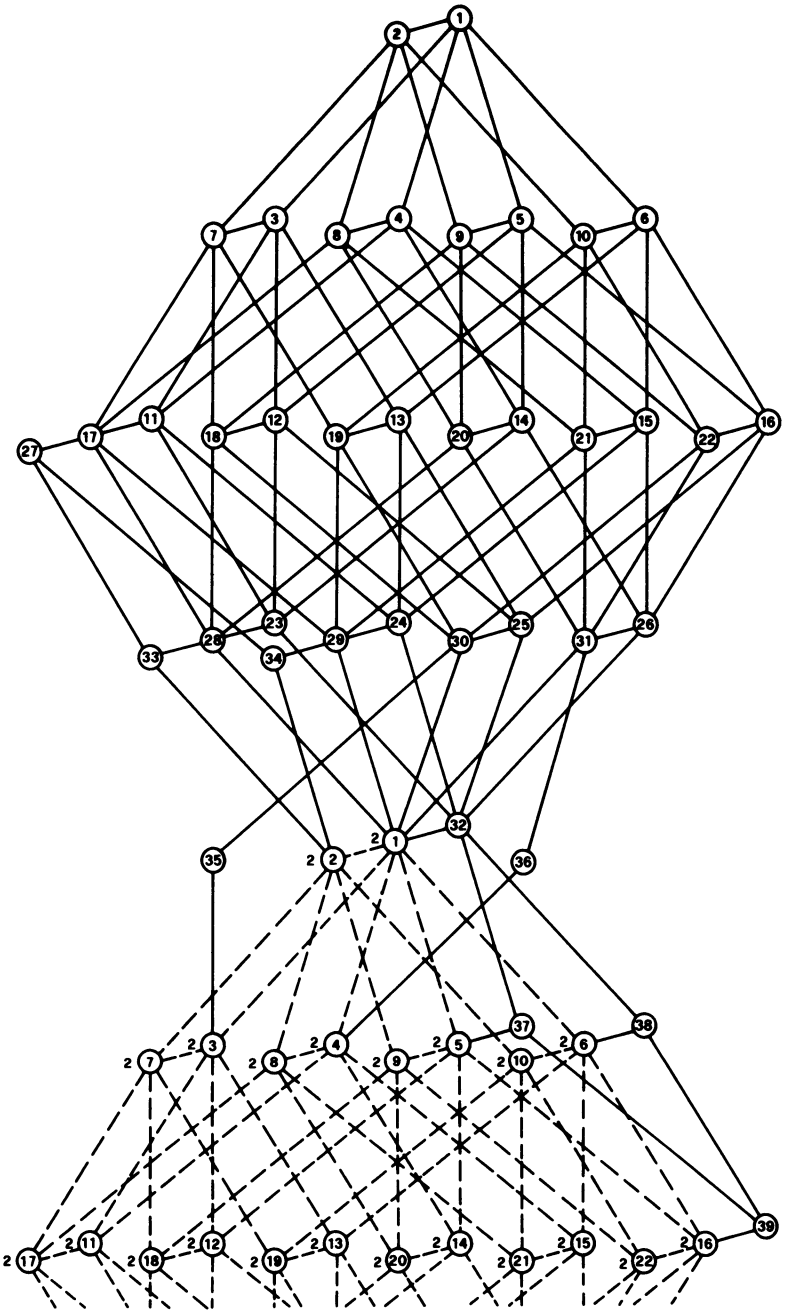
(3) $B(M_7) = I_3 \otimes (J_3 - I_3)$, $B(M_i)$ $(i = 8, 9, 10)$ are obtained as in (6),

(8) $B(M_{11}) = (-e_1 + e_2 + e_3, e_1 - e_2 + e_3, e_1 + e_2 - e_3, -e_4 + e_5 + e_6, e_4 - e_5 + e_6, e_4 + e_5 - e_6, e_1 + e_5 + e_9, e_2 + e_6 + e_7, e_3 + e_4 + e_8)$, for $B(M_i)$ $(i = 12, \ldots, 16)$ compare (6),

$$(9) \quad B(M_{17}) = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & -1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}, \quad \text{for } B(M_i)\ (i = 18, \ldots, 22) \text{ compare (6)},$$

(10) $B(M_{27}) = D((123)(798))((J_3 - I_3) \otimes (J_3 - I_3))$,

(5) $B(M_{26}) = I_3 \otimes (J_3 - 2I_3)$, for $B(M_i)$ $(i = 23, 24, 25)$ compare (6),



(Read $\textcircled{i} = M_i$ $(i = 1, \ldots, 39)$.)

(7) $B(M_{31}) = (2e_1, 2e_2, 2e_3, 2e_4, 2e_5, 2e_7, 2e_8, x_3, y_3)$ with $x_3^T = (0, 0, 0, 1, 1, 1, 1, 1, 1)$, $y_3^T = (1, 1, 1, 0, 0, 0, 1, 1, -1)$, for $B(M_i)$ $(i = 28, 29, 30)$ compare (6),

(3) $B(M_{32}) = B(L_7)$,

(12) $B(M_{33}) = B(L_8)$, $B(M_{34}) = T_1 \cdot B(M_{33})$ for a suitable monomial matrix T_1,

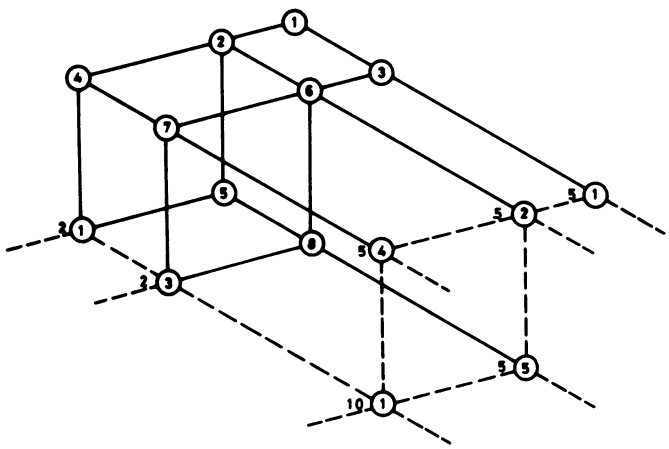
$$B(M_{36}) = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix} \otimes (J_3 - 2I_3), \tag{14}$$

$$B(M_{35}) = T_2 \cdot B(M_{36}) \quad \text{for a suitable monomial matrix } T_2,$$

$$B(M_{38}) = B(L_9), \tag{13}$$

$$B(M_{37}) = T_3 \cdot B(M_{38}) \quad \text{for a suitable monomial matrix } T_3.$$

At last we describe the $\prec$-maximal centerings of G_3: $N_1^\# = N_6$, $N_2^\# = N_3$, $N_4^\# = N_8$, $N_5^\# = N_7$.



(Read $\textcircled{i} = N_i$ $(i = 1, \ldots, 8)$.)

(15) $B(N_1) = I_9$,

(17) $B(N_2) = (e_1 + e_2, e_1 + e_3, \ldots, e_1 + e_9, -e_2 - e_3)$,

(18) $B(N_3) = (-e_1 + e_2, -e_1 + e_3, \ldots, -e_1 + e_9, e_5 + e_6 + e_7 + e_8 + e_9)$,

$$B(N_4) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}, \quad B(N_5) = D((1475)(2698))B(N_4), \tag{19}$$

(16) $B(N_6) = I_9 + J_9$,

$$B(N_7) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \\ 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 1 \\ -1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & -1 & -1 & -1 & 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & -1 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 & 0 & -1 & -1 & 0 & 1 \end{pmatrix}, \tag{20}$$

$$B(N_8) = D((1475)(2698))B(N_7).$$

4. The Irreducible Maximal Finite Subgroups of $GL(9, \mathbf{Z})$. There are—up to $\mathbf{Z}$-equivalence—20 such groups. They fall into eight $\mathbf{Q}$-classes. Fourteen of these groups (belonging to six $\mathbf{Q}$-classes) are rationally equivalent to a monomial group and nine (belonging to three $\mathbf{Q}$-classes) are nonsolvable.

(4.1) THEOREM. *The irreducible maximal finite subgroups of $GL(9, \mathbf{Z})$ are $\mathbf{Z}$-equivalent to the $\mathbf{Z}$-automorphism groups of the quadratic forms $F_1, \ldots, F_{20}$.*

(i) $\mathrm{Aut}(F_1)$, $\mathrm{Aut}(F_2)$, *and* $\mathrm{Aut}(F_3)$ *are* $\mathbf{Q}$-*equivalent. They are isomorphic to the wreath product* $C_2 \sim S_9$ *of order* $2^9 9!$. $\mathrm{Aut}(F_1)$ *is the full monomial group of degree* 9.

(ii) $\mathrm{Aut}(F_4)$, $\mathrm{Aut}(F_5)$, $\mathrm{Aut}(F_6)$, *and* $\mathrm{Aut}(F_7)$ *are* $\mathbf{Q}$-*equivalent and isomorphic to the wreath product* $(C_2 \sim S_3) \sim S_3$ $(\cong (C_2 \times S_4) \sim S_3)$ *of order* $(2^3 3!)^3 3!$.

(iii) $\mathrm{Aut}(F_8) \sim_{\mathbf{Q}} \mathrm{Aut}(F_9)$. *Both are isomorphic to a split extension of an elementary abelian group of order* 2^9 *by a wreath product* $(S_3 \sim C_2)$. *The order is* $2^9(3!)^2 2$.

(iv) $\mathrm{Aut}(F_{10}) \sim_{\mathbf{Q}} \mathrm{Aut}(F_{11})$. *They are isomorphic to* $C_2 \times (S_4 \sim C_2)$ *of order* $2(4!)^2 2$. (*The groups can be considered as crown products of* $C_2 \times S_4$ *by* C_2.)

(v) $\mathrm{Aut}(F_{12}) \sim_{\mathbf{Q}} \mathrm{Aut}(F_{13})$. *The groups are isomorphic to* $C_2 \times (S_4 \sim S_3)$ *of order* $2(4!)^3 3!$.

(vi) $\mathrm{Aut}(F_{14})$ *is isomorphic to* $C_2 \times S_4 \times S_4$ *of order* $2(4!)^2$.

(vii) $\mathrm{Aut}(F_{15})$, $\mathrm{Aut}(F_{16})$, $\mathrm{Aut}(F_{17})$, *and* $\mathrm{Aut}(F_{18})$ *are* $\mathbf{Q}$-*equivalent and isomorphic to* $C_2 \times S_{10}$ *of order* $2(10!)$.

(viii) $\mathrm{Aut}(F_{19}) \sim_{\mathbf{Q}} \mathrm{Aut}(F_{20})$. *Both are isomorphic to* $C_2 \times S_6$ *of order* $2(6!)$.

For a better understanding of the cases (ii), (iv), and (v) we note that the irreducible maximal finite subgroups of $GL(3, \mathbf{Z})$ are all rationally equivalent and isomorphic to $C_2 \sim S_3$ $(\cong C_2 \times S_4)$.

Proof. Ad(i). Compare Theorems (6.1), (6.2) in Part I [7].

Ad(ii). We have $4F_4^{-1} = F_5$ and $4F_6^{-1} \sim_{\mathbf{Z}} F_7$. Hence, by transposing the matrices of $\mathrm{Aut}_{\mathbf{Z}}(F_4)$ $(\mathrm{Aut}_{\mathbf{Z}}(F_6))$ one obtains a group which is $\mathbf{Z}$-equivalent to $\mathrm{Aut}_{\mathbf{Z}}(F_5)$ $(\mathrm{Aut}_{\mathbf{Z}}(F_7))$. Since $F_4 = (I_3 + J_3) \oplus (I_3 + J_3) \oplus (I_3 + J_3)$, $\mathrm{Aut}_{\mathbf{Z}}(F_4)$ is the wreath

product of $\mathrm{Aut}_{\mathbf{Z}}(I_3 + J_3)$ with S_3 [2], $\mathrm{Aut}_{\mathbf{Z}}(I_3 + J_3)$ being isomorphic to $C_2 \sim S_3$. It remains to show that $\mathrm{Aut}_{\mathbf{Z}}(F_4)$ and $\mathrm{Aut}_{\mathbf{Z}}(F_6)$ are rationally equivalent. In the lattice of centerings for G_2 the centerings M_7, M_3 belong to the forms F_4, F_6, respectively. Let G be the biggest subgroup of $\mathrm{Aut}(M_7)$ which fixes the quadratic form belonging to M_7 (i.e. $G \cong \mathrm{Aut}_{\mathbf{Z}}(F_4)$). We prove that G is rationally equivalent to a subgroup of $\mathrm{Aut}_{\mathbf{Z}}(F_6)$ by showing that G leaves M_3 invariant. Clearly, G leaves $M_{26} = M_7^{\#}$ invariant and, hence, also $M_1 = M_7 + M_7^{\#}$. But even the maximal finite subgroup of $GL(9, \mathbf{Z})$ belonging to M_1 and F_1 (i.e. the full monomial group of degree 9) leaves M_{32} invariant. Hence, G leaves $M_3 = M_7 + M_{32}$ invariant. Similarly, one sees that $\mathrm{Aut}_{\mathbf{Z}}(F_6)$ is rationally equivalent to a subgroup of $\mathrm{Aut}_{\mathbf{Z}}(F_4)$ (note $M_7 = M_2 \cap M_3$).

Ad(iii). The forms $4F_8^{-1}$ and F_9 are integrally equivalent. Therefore, the corresponding automorphism groups are $\mathbf{Q}$-equivalent. We determine $\mathrm{Aut}_{\mathbf{Z}}(F_8)$. In the lattice of centerings for G_2 for instance M_{11} belongs to F_8; hence, the biggest subgroup G of $\mathrm{Aut}(M_{11})$ fixing the corresponding form is isomorphic to $\mathrm{Aut}_{\mathbf{Z}}(F_8)$. G also leaves $M_{11}^{\#} = M_{22}$, hence, $M_1 = M_{11} + M_{22}$ invariant. Thus, G is $\mathbf{Q}$-equivalent to a subgroup $\bar{G}$ of the full monomial group, more precisely to the biggest subgroup of $\mathrm{Aut}(M_1)$ which leaves M_{11} and the form belonging to M_1 invariant. Clearly, $\bar{G}$ contains the minimal irreducible group G_2, if we identify $\mathrm{Aut}(M_1)$ with $GL(9, \mathbf{Z})$ via the canonical basis. Furthermore, $\bar{G}$ contains all diagonal matrices of $GL(9, \mathbf{Z})$, since they fix $M_1/2M_1$ pointwise. It follows that the diagonal matrices form a normal subgroup of $\bar{G}$ of order 2^9, having a transitive permutation group P of degree 9 as complement. Considering the vectors of shortest length in M_{11} we find that P has to be the maximal subgroup of S_9 permuting the sets $\{1, 2, 3\}$, $\{4, 5, 6\}$, $\{7, 8, 9\}$, $\{1, 5, 9\}$, $\{2, 6, 7\}$, $\{3, 4, 8\}$. By a short computation P acts faithful and imprimitive on these six sets and is isomorphic to $S_3 \sim C_2$.

Ad(iv). The forms $16F_{10}^{-1}$ and F_{11} are integrally equivalent, hence, $\mathrm{Aut}_{\mathbf{Z}}(F_{10}) \sim_{\mathbf{Q}} \mathrm{Aut}_{\mathbf{Z}}(F_{11})$. We determine the automorphism group of $F_{10} = (I_3 + J_3) \otimes (I_3 + J_3)$. Clearly, $\mathrm{Aut}_{\mathbf{Z}}(F_{10})$ contains all $g \otimes h$ with $g, h \in \mathrm{Aut}_{\mathbf{Z}}(I_3 + J_3) \cong C_2 \times S_4$ as well as an involution interchanging the components of the tensor product. But these elements already generate $\mathrm{Aut}_{\mathbf{Z}}(F_{10})$. This is shown by similar arguments as in (ii) or (iii); in particular, it follows easily that $\mathrm{Aut}_{\mathbf{Z}}(F_{10})$ is rationally equivalent to a subgroup of $\mathrm{Aut}_{\mathbf{Z}}(F_8)$. The index can be computed as in case (v) below.

Ad(v). Again the forms $16F_{12}^{-1}$ and F_{13} are integrally equivalent; hence, $\mathrm{Aut}_{\mathbf{Z}}(F_{12}) \sim_{\mathbf{Q}} \mathrm{Aut}_{\mathbf{Z}}(F_{13})$. We determine $\mathrm{Aut}_{\mathbf{Z}}(F_{12})$. In the lattice of centerings for G_2 the centering M_{33} belongs to F_{12}. Let G be the biggest subgroup of $\mathrm{Aut}(M_{33})$ fixing the corresponding form, $G \cong \mathrm{Aut}_{\mathbf{Z}}(F_{12})$. Then G leaves $M_{23} = M_{33} + M_{33}^{\#}$ invariant. Hence, G is isomorphic to a subgroup of $\mathrm{Aut}_{\mathbf{Z}}(F_5) \cong (C_2 \times S_4) \sim S_3$. Let $\bar{G}$ be the maximal subgroup of $\mathrm{Aut}(M_{33})$ fixing F_5. The group $\bar{G}$ acts on $\{gM_{33} | g \in \bar{G}\} =: S$, transitively. The set S consists of four elements. Then G is isomorphic to the stabilizer of M_{33} in $\bar{G}$. The latter turns out to be isomorphic to $C_2 \times (S_4 \sim S_3)$.

Ad(vi). The automorphism group of the form $F_{14} = (I_3 + J_3) \otimes (4I_3 - J_3)$ certainly contains the elements $g \otimes h$ with $g \in \mathrm{Aut}_{\mathbf{Z}}(I_3 + J_3) \cong C_2 \times S_4$, $h \in$

$\mathrm{Aut}_{\mathbf{Z}}(4I_3 - J_3) \cong C_2 \times S_4$. These form already the full automorphism group as can be seen similar to the previous cases.

Ad(vii). It is $10F_{15}^{-1} = F_{16}$ and $10F_{17}^{-1} \sim_{\mathbf{Z}} F_{18}$. Therefore, $\mathrm{Aut}_{\mathbf{Z}}(F_{15}) \sim_{\mathbf{Q}} \mathrm{Aut}_{\mathbf{Z}}(F_{16})$ and $\mathrm{Aut}_{\mathbf{Z}}(F_{17}) \sim_{\mathbf{Q}} \mathrm{Aut}_{\mathbf{Z}}(F_{18})$. Since the representation module N_1 of G_3 has—up to sign—only ten vectors of minimal length with not any two of them being orthogonal, the group $\mathrm{Aut}_{\mathbf{Z}}(F_{15})$ is easily seen to be isomorphic to $C_2 \times S_{10}$ (compare also [6]). To prove that $\mathrm{Aut}_{\mathbf{Z}}(F_{15})$ is rationally equivalent to a subgroup of $\mathrm{Aut}_{\mathbf{Z}}(F_{17})$ we consider the lattice of centerings of $\mathrm{Aut}_{\mathbf{Z}}(F_{15})$ viewed as a subgroup of $\mathrm{Aut}(N_1)$. It consists of the multiples of N_1, $N_8 = N_1^{\#}$, $N_2 = N_1^{\#} + 2N_1$, $N_3 = N_2^{\#}$ ([6], [7]). Therefore, N_2 is invariant under $\mathrm{Aut}_{\mathbf{Z}}(F_{15})$. On the other hand $N_1 = N_2 + N_2^{\#}$ holds and $\mathrm{Aut}_{\mathbf{Z}}(F_{17})$ is rationally equivalent to a subgroup of $\mathrm{Aut}_{\mathbf{Z}}(F_{15})$, hence to $\mathrm{Aut}_{\mathbf{Z}}(F_{15})$ itself.

Ad(viii). The forms $20F_{19}^{-1}$ and F_{20} are integrally equivalent; and hence, $\mathrm{Aut}_{\mathbf{Z}}(F_{19}) \sim_{\mathbf{Q}} \mathrm{Aut}_{\mathbf{Z}}(F_{20})$. In the lattice of centerings for G_3 the centering N_5 belongs to the form F_{19}. Because of $N_5 + N_5^{\#} = N_2$ and $N_2 + N_2^{\#} = N_1$ the group $\mathrm{Aut}_{\mathbf{Z}}(F_{19})$ is rationally equivalent to a subgroup of $\mathrm{Aut}_{\mathbf{Z}}(F_{15}) \cong C_2 \times S_{10}$. Therefore, $\mathrm{Aut}_{\mathbf{Z}}(F_{19})$ is isomorphic to $C_2 \times P$, where P is a subgroup of S_{10} maximal with the properties:

(α) P contains $PSL(2, 9)$ acting on the ten points of the projective line over $\mathbf{F}_9$; compare Lemma (2.3).

(β) The natural permutation representation of P has two constituents unequal 1 modulo 2, each of degree 4.

From the list of primitive permutation groups up to degree 20 [9] P has to be a subgroup of $\mathrm{Aut}(PSL(2, 9)) \cong P\Gamma L(2, 9)$, since A_{10} and S_{10} violate property (β). The decomposition numbers modulo 2 of S_6 show that the subgroup of $P\Gamma L(2, 9)$ which is isomorphic to S_6 still fulfills (α) and (β) [5]. To prove $S_6 \cong P$ it remains to show that (β) does not hold for $P\Gamma L(2, 9)$. But (β) implies that P is isomorphic to a subgroup of $GL(4, 2)$, and the assumption $P = P\Gamma L(2, 9)$ would lead to a primitive permutation representation of $GL(4, 2)$ of degree $|GL(4, 2)|/|P\Gamma L(2, 9)| = 14$ which does not exist by inspection of Table I in [9]. Q.E.D.

RWTH Aachen
Lehrstuhl D für Mathematik
Templergraben 64
5100 Aachen, West Germany

Mathematisches Institut der Universität zu Köln
Weyertal 86–90
5000 Köln 41, West Germany

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