

## Corrigendum to "What Drives an Aliquot Sequence?"

By Richard K. Guy and J. L. Selfridge

**Abstract.** An aliquot sequence  $n : k$ ,  $k = 0, 1, 2, \dots$ , is defined by  $n : 0 = n$ ,  $n : k + 1 = \sigma(n : k) - n : k$ , and a driver of an aliquot sequence is a number  $2^A v$  with  $A > 0$ ,  $v$  odd,  $v | 2^{A+1} - 1$  and  $2^{A-1} | \sigma(v)$ . Pollard has noted some errors in a proof in [1] that the drivers comprise the even perfect numbers and a finite set. These are now corrected in a revised proof.

John Pollard has observed two inaccuracies and some obscurities in a proof in [1] for which we wish to substitute the following.

**THEOREM 2.** *The only drivers are 2, 2<sup>3</sup>3, 2<sup>3</sup>3.5, 2<sup>5</sup>3.7, 2<sup>9</sup>3.11.31 and the even perfect numbers.*

*Proof.* A driver is  $2^A v$  with  $A > 0$ ,  $v$  odd,  $v | 2^{A+1} - 1$  and  $2^{A-1} | \sigma(v)$ . If  $v = 1$ ,  $2^{A-1} | 1$ ,  $A = 1$  and we have the "downdriver" 2. If  $v = 2^{A+1} - 1$  is a Mersenne prime, the driver is an even perfect number. Henceforth, we assume that  $v > 1$  and that  $2^{A+1} - 1$  is composite.

If  $p^a || 2^{A+1} - 1$ ,  $p$  prime,  $a > 0$ , define the *deficiency*,  $\delta(p)$ , of  $p$  to be  $2^d/p^a$ , where  $2^d || \sigma(p^b)$  and  $p^b || v$ ,  $0 \leq b \leq a$ . The product of all the deficiencies is greater than  $1/4$ , since otherwise

$$2^{A+1} > 2^{A+1} - 1 = \prod_p p^a \geq 4 \prod_a 2^d,$$

$2^{A-1} > \prod 2^d$  and  $2^{A-1}$  would not divide  $\prod \sigma(p^b) = \sigma(v)$ .

The power of 2 dividing  $\sigma(p^b)$  depends only on how many factors of the product  $(p+1)(p^2+1)(p^4+1) \dots$  divide  $\sigma(p^b)$ , each factor other than  $p+1$  contributing a single 2. Hence,  $d = 0$  if  $b$  is even and  $d = t + k - 1$  if  $b$  is odd, where  $2^t || p + 1$ , there are  $k$  such factors, and thus  $2^k || b + 1$ . It then follows that

$$\delta(p) \leq (p+1)(b+1)/2p^a \leq (p+1)(a+1)/2p^a.$$

If  $p$  is a Mersenne prime and  $a = b = 1$ ,  $\delta(p) = (p+1)/p > 1$ . Otherwise,  $\delta(p) < 1$ . If  $p$  is not a Mersenne prime, then  $\delta(p) \leq 2/5$  ( $\delta(5) = 2/5$  if  $a = b = 1$ ),  $\delta(p) \leq 4/11$  if  $p > 5$ , and  $\delta(p) \leq 2/25$  if  $a \geq 2$ . If we denote by  $\prod \delta(p)$  the product of the deficiencies of the Mersenne prime factors of  $2^{A+1} - 1$ , it is not difficult to see that

$$\prod \delta(p) \leq \frac{4}{3} \cdot \frac{8}{7} \cdot \frac{32}{31} \cdot \frac{128}{127} \dots < \frac{4}{3} \cdot \frac{8}{7} \cdot \frac{32}{31} \cdot \frac{64}{63} < \frac{8}{5}.$$

We now note that  $2^{A+1} - 1$  contains at most one non-Mersenne prime factor.

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For having two such prime factors would imply that the product of the deficiencies would be less than

$$\delta(p_1)\delta(p_2) \prod \delta(p) < \frac{2}{5} \cdot \frac{4}{11} \cdot \frac{8}{5} < \frac{1}{4},$$

while  $p_1^2 | 2^{A+1} - 1$  is impossible since

$$\delta(p_1) \prod \delta(p) < \frac{2}{25} \cdot \frac{8}{5} < \frac{1}{4}.$$

For a Mersenne prime  $2^a - 1 > 7$ ,  $a > 1$  would imply  $\delta(2^a - 1) \leq 32/31^2$ . But  $(32/31^2)(8/5) < 1/4$ . For  $p = 7$ ,  $a > 1$  would imply

$$\delta(7) \prod_{p \neq 7} \delta(p) < \frac{8}{7^2} \cdot \frac{7}{5} < \frac{1}{4}.$$

For  $p = 3$ ,  $a > 3$  would imply  $\delta(3) \leq 8/81$ . But  $(8/81)(8/5) < 1/4$ .

If  $p^a = 3^3$ ,  $3^3 | 2^{A+1} - 1$ ,  $18 | A + 1$ ,  $19.73 | 2^{A+1} - 1$ . But neither 19 nor 73 is a Mersenne prime: contradiction. If  $p^a = 3^2$ ,  $6 | A + 1$ . If  $A = 5$  we have the driver  $2^5 3.7$ , while for odd  $A > 5$ ,  $2^{A+1} - 1$  contains a non-Mersenne prime factor  $p_1$  and

$$\delta(3)\delta(p_1) \prod_{p \neq 3} \delta(p) < \frac{4}{9} \cdot \frac{2}{5} \cdot \frac{6}{5} < \frac{1}{4}.$$

If  $2 \leq q_1 < \dots < q_k$ , then  $2^{A+1} - 1 = (2^{q_1} - 1) \dots (2^{q_k} - 1)$  is impossible modulo  $2^{q_1+1}$ , and we have only to consider

$$2^{A+1} - 1 = (2^{q_1} - 1) \dots (2^{q_k} - 1)(2^c u - 1), \quad u \text{ odd}, u \geq 3.$$

We know that  $u = 3$  or  $5$ , since  $u \geq 7$  would imply

$$\delta(2^c u - 1) \prod \delta(p) < \frac{2}{13} \cdot \frac{8}{5} < \frac{1}{4}.$$

If  $c = 1$ ,  $u = 3$  (since  $2.5 - 1$  is not prime),  $2u - 1 = 5$ ,  $5 | 2^{A+1} - 1$ ,  $A + 1 = 4k$ ,  $15 | 2^{A+1} - 1$ . If  $A = 3$ , we have the drivers  $2^3 3$  and  $2^3 3.5$ , while if  $A \geq 7$ , there is a prime  $p$ ,  $p | 2^{A+1} - 1$ ,  $p \equiv 1 \pmod{A + 1}$ , giving a second non-Mersenne prime divisor of  $2^{A+1} - 1$ .

So we have  $c \geq 2$ ,  $q_1 \geq 2$ ,  $u = 3$  or  $5$  and

$$-1 \equiv (2^{q_1} - 1)(-1) \dots (-1)(2^c u - 1) \pmod{2^{\min(c, q_1)+1}},$$

$-1 \equiv (-1)^{k-1}(-2^{q_1} - 2^c u + 1)$ ,  $k$  is even and  $q_1 = c$ . Now  $2^{A+1} < 2^{q_1} \dots 2^{q_k} 2^c u$  and  $2^q - 1$  divides  $2^{A+1} - 1$  only if  $q | A + 1$  and the  $q_i$  are distinct primes. Therefore,

$$q_1 \dots q_k | A + 1 < q_1 + \dots + q_k + c + \log_2 u < q_1 + \dots + q_k + q_1 + 3.$$

If  $k \geq 3$ , this would imply  $2.3.q_3 \leq q_1 q_2 q_3 < 2q_1 + q_2 + q_3 + 3 < 4q_3 + 3$ , a contradiction. So  $k = 2$ ,  $q_1 q_2 < 2q_1 + q_2 + 3$ ,  $(q_1 - 1)(q_2 - 2) < 5$ ,  $q_1 = 2 = c$  and  $q_2 = 3$  or  $5$ . Only the latter gives a solution;  $u = 3$  and  $2^9 3.11.31$  is a driver.

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1. RICHARD K. GUY & J. L. SELFRIDGE, "What drives an aliquot sequence?," *Math. Comp.*, v. 29, 1975, pp. 101–107. MR 52 #5542.