Corrigendum to “What Drives an Aliquot Sequence?”

By Richard K. Guy and J. L. Selfridge

Abstract. An aliquot sequence \( n : k, k = 0, 1, 2, \ldots \), is defined by \( n : 0 = n, n : k + 1 = o(n : k) - n : k \), and a driver of an aliquot sequence is a number \( 2^A v \) with \( A > 0 \), \( v \) odd, \( v | 2^{A+1} - 1 \) and \( 2^{A-1} | o(v) \). Pollard has noted some errors in a proof in [1] that the drivers comprise the even perfect numbers and a finite set. These are now corrected in a revised proof.

John Pollard has observed two inaccuracies and some obscurities in a proof in [1] for which we wish to substitute the following.

**Theorem 2.** The only drivers are \( 2, 2^3, 2^3.5, 2^5.3.7, 2^9.3.11.31 \) and the even perfect numbers.

**Proof.** A driver is \( 2^A v \) with \( A > 0 \), \( v \) odd, \( v | 2^{A+1} - 1 \) and \( 2^{A-1} | o(v) \). If \( v = 1 \), \( 2^{A-1} | 1 \), \( A = 1 \) and we have the “downdriver” \( 2 \). If \( v = 2^{A+1} - 1 \) is a Mersenne prime, the driver is an even perfect number. Henceforth, we assume that \( v > 1 \) and that \( 2^{A-1} \) is composite.

If \( p^a \| 2^{A+1} - 1 \), \( p \) prime, \( a > 0 \), define the deficiency, \( \delta(p) \), of \( p \) to be \( 2^d/p^a \), where \( 2^d \| o(p^b) \) and \( p^b \| v \), \( 0 \leq b \leq a \). The product of all the deficiencies is greater than \( 1/4 \), since otherwise

\[
2^{A+1} > 2^{A+1} - 1 = \prod_p p^a \geq 4 \prod_d 2^d,
\]

\( 2^{A-1} > \prod 2^d \) and \( 2^{A-1} \) would not divide \( \prod o(p^b) = o(v) \).

The power of 2 dividing \( o(p^b) \) depends only on how many factors of the product \( (p + 1)(p^2 + 1)(p^4 + 1) \ldots \) divide \( o(p^b) \), each factor other than \( p + 1 \) contributing a single 2. Hence, \( d = 0 \) if \( b \) is even and \( d = t + k - 1 \) if \( b \) is odd, where \( 2^t \| p + 1 \), there are \( k \) such factors, and thus \( 2^k \| b + 1 \). It then follows that

\[
\delta(p) \leq (p + 1)(b + 1)/2p^a \leq (p + 1)(a + 1)/2p^a.
\]

If \( p \) is a Mersenne prime and \( a = b = 1 \), \( \delta(p) = (p + 1)/p > 1 \). Otherwise, \( \delta(p) < 1 \). If \( p \) is not a Mersenne prime, then \( \delta(p) \leq 2/5 \) (\( \delta(5) = 2/5 \) if \( a = b = 1 \), \( \delta(p) \leq 4/11 \) if \( p > 5 \), and \( \delta(p) \leq 2/25 \) if \( a > 2 \). If we denote by \( \Pi \delta(p) \) the product of the deficiencies of the Mersenne prime factors of \( 2^{A+1} - 1 \), it is not difficult to see that

\[
\Pi \delta(p) \leq \frac{4}{3} \cdot \frac{8}{7} \cdot \frac{32}{31} \cdot \frac{128}{127} \ldots \leq \frac{4}{3} \cdot \frac{8}{7} \cdot \frac{32}{31} \cdot \frac{64}{63} \leq \frac{8}{5}.
\]

We now note that \( 2^{A+1} - 1 \) contains at most one non-Mersenne prime factor.

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For having two such prime factors would imply that the product of the deficiencies would be less than

\[ \delta(p_1)\delta(p_2) \prod \delta(p) < \frac{2}{5} \cdot \frac{4}{11} \cdot \frac{8}{5} < \frac{1}{4}, \]

while \( p_1^2 | 2^{A+1} - 1 \) is impossible since

\[ \delta(p_1) \prod \delta(p) < \frac{2}{25} \cdot \frac{8}{5} < \frac{1}{4}. \]

For a Mersenne prime \( 2^q - 1 > 7, a > 1 \) would imply \( \delta(2^q - 1) \leq 32/31^2 \). But \((32/31^2)(8/5) < 1/4\). For \( p = 7 \), \( a > 1 \) would imply

\[ \delta(7) \prod_{p \neq 7} \delta(p) < \frac{8}{7^2} \cdot \frac{7}{5} < \frac{1}{4}. \]

For \( p = 3, a > 3 \) would imply \( \delta(3) \leq 8/81 \). But \((8/81)(8/5) < 1/4\).

If \( p^a = 3^3, 3^3 \mid 2^{A+1} - 1, 18 \mid A + 1, 19.73 \mid 2^{A+1} - 1 \). But neither 19 nor 73 is a Mersenne prime: contradiction. If \( p^a = 3^2, 6 \mid A + 1 \). If \( A = 5 \) we have the driver \( 2^5.3.7 \), while for odd \( A > 5, 2^{A+1} - 1 \) contains a non-Mersenne prime factor \( p_1 \) and

\[ \delta(3) \delta(p_1) \prod_{p \neq 3} \delta(p) < \frac{4}{9} \cdot \frac{2}{5} \cdot \frac{6}{5} < \frac{1}{4}. \]

If \( 2 < q_1 < \cdots < q_k \), then \( 2^{A+1} - 1 = (2^q_1 - 1) \cdots (2^q_k - 1) \) is impossible modulo \( 2^{q_1 + 1} \), and we have only to consider

\[ 2^{A+1} - 1 = (2^q_1 - 1) \cdots (2^q_k - 1)(2^c u - 1), \quad u \text{ odd, } u \geq 3. \]

We know that \( u = 3 \) or 5, since \( u \geq 7 \) would imply

\[ \delta(2^c u - 1) \prod \delta(p) < \frac{2}{13} \cdot \frac{8}{5} < \frac{1}{4}. \]

If \( c = 1, u = 3 \) (since \( 2.5 - 1 \) is not prime), \( 2u - 1 = 5, 5 \mid 2^{A+1} - 1, A + 1 = 4k, 15 \mid 2^{A+1} - 1 \). If \( A = 3 \), we have the drivers \( 2.3 \cdot 3.5 \) and \( 2^3.3.5 \), while if \( A \geq 7 \), there is a prime \( p, p \mid 2^{A+1} - 1, p \equiv 1 \mod (A + 1) \), giving a second non-Mersenne prime divisor of \( 2^{A+1} - 1 \).

So we have \( c \geq 2, q_1 \geq 2, u = 3 \) or 5 and

\[-1 \equiv (2^q_1 - 1)(-1) \cdots (-1)(2^c u - 1) \mod 2^{\min(c,q_1)+1},
\]

\[-1 \equiv (-1)^{k-1}(-2^q - 1 - 2^c u + 1), \quad k \text{ is even and } q_1 = c. \]

Now \( 2^{A+1} < 2^q_1 \cdots 2^q_k 2^c u \) and \( 2^q - 1 \) divides \( 2^{A+1} - 1 \) only if \( q \mid A + 1 \) and the \( q_i \) are distinct primes. Therefore,

\[ q_1 \cdots q_k \mid A + 1 < q_1 + \cdots + q_k + c + \log_2 u < q_1 + \cdots + q_k + q_1 + 3. \]

If \( k \geq 3, \) this would imply \( 2.3.3^3 < q_1 q_2 q_3 < 2q_1 + q_2 + q_3 + 3 < 4q_3 + 3 \), a contradiction. So \( k = 2, q_1 q_2 < 2q_1 + q_2 + 3, (q_1 - 1)q_2 - 2 < 5, q_1 = 2 = c \) and \( q_2 = 3 \) or 5. Only the latter gives a solution; \( u = 3 \) and \( 2^9.3.11.31 \) is a driver.
CORRIGENDUM TO "WHAT DRIVES AN ALIQUOT SEQUENCE?"

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