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Convergence of Multi-Grid Iterations Applied to Difference Equations

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Abstract. Convergence proofs for the multi-grid iteration are known for the case of finite element equations and for the case of some difference schemes discretizing boundary value problems in a rectangular region. In the present paper we give criteria of convergence that apply to general difference schemes for boundary value problems in Lipschitzian regions. Furthermore, convergence is proved for the multi-grid algorithm with Gauss-Seidel's iteration as smoothing procedure.

1. Introduction. Systems of linear equations arising from boundary value problems can be solved very fast by the multi-grid iteration (cf. [1]–[6], [9], [11]). Although, the multi-grid algorithms are applied successfully to a general class of problems, the proofs of convergence are restricted to a very special class of problems. In the case of special finite element equations for boundary value problems with smooth boundaries proofs of convergence are given by Astrachancev [1] and Nicolaides [9]. In [6] the author established general criteria and proved the convergence for general finite element problems.

The second important class of problems are systems of difference equations discretizing boundary value problems. The model problem of Poisson's equation in a rectangle (and similar problems) can be analyzed easily by means of Fourier transformation (cf. Fedorenko [4]). In the case of certain difference schemes for problems with variable coefficients and a rectangular region, Bachvalov [2] and Wesseling [11] proved the convergence of the multi-grid iteration. But two gaps are still to be filled. Convergence proofs are missing for the case of nonrectangular regions. Moreover, all proofs cited above require a special smoothing procedure (cf. Section 4) related to the Jacobi iteration. In practice smoothing by the Gauss-Seidel iteration is preferred (cf. [3], [5]). This paper contains general criteria that apply to difference schemes in general regions and to smoothing by Gauss-Seidel.

In Section 2 we describe the multi-grid algorithm very briefly. For further comments we refer, for instance, to [6]. As pointed out in [6] the convergence can be concluded from an 'approximation property' and a 'smoothing property'. The first one is studied in Section 3. A criterion is proved and its assumptions are verified in the case of a very general difference scheme. It turns out that the crux of the assumptions is a certain regularity condition (3.6b) that is proved in [7] for the case of Dirichlet

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boundary values. The smoothing property is investigated in Section 4, in particular, for the case of Gauss-Seidel's iteration as smoothing procedure.

2. Multi-Grid Iteration. Let

$$h_0 > h_1 > \cdots > h_l > \cdots > 0 \tag{2.1}$$

be a sequence of grid sizes. l is called the 'level number'. The discretization of the continuous problem (boundary value problem)

$$Lu = f \tag{2.2}$$

with step size h_l is denoted by

$$L_l u_l = f_l \qquad (l \geqslant 0). \tag{2.3}$$

The solution u_l of (2.3), as well as the right-hand side f_l, belongs to a finite-dimensional vector space V_l.

The system (2.3) of linear equations is to be solved by the multi-grid algorithm described below. It uses auxiliary equations of the form $L_m u_m = g_m$ for $m = 0, 1, \ldots, l-1$. The connection of grid functions of different levels is given by a prolongation

$$p_{l,l-1}\colon\ V_{l-1} \longrightarrow V_l$$

and a restriction

$$r_{l-1,l}\colon\ V_l \longrightarrow V_{l-1}.$$

Since a detailed explanation of the multi-grid algorithm is contained in [5], [6], we give only a brief description by means of a program.

```
procedure mgm(l, u, f): integer l; array u, f;
if l = 0 then u := L_0^{-1} * f_0 else
begin integer j; array v, d;
    for j := 1 step 1 until ν do u := G_l(u, f):
    d := r_{l-1,l} * (L_l * u - f); v := 0;
    for j := 1 step 1 until γ do mgm(l - 1, v, d);
    u := u - p_{l,l-1} * v
end;
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The meaning of the parameters is the following. $l \geqslant 0$ is the actual level number. $f \in V_l$ is the right-hand side to the problem in consideration (e.g., $f = f_l$ in case of (2.3)). u has an arbitrary input value $u_l^{(i)} \in V_l$ (i: number of iterations). The procedure *mgm* computes the next iterate $u = u_l^{(i+1)}$ as output. The procedure depends on the positive numbers ν (number of iterations of the smoothing procedure G_l) and γ (number of *mgm* iterations per level). The smoothing procedure is of the form

$$G_l(v_l, f_l) = G_l v_l + H_l f_l \quad (v_l, f_l \in V_l) \text{ with } G_l + H_l L_l = I. \tag{2.4}$$

The convergence of the multi-grid algorithm depends on the choice of ν, γ, on the coarsest step size h_0 and on the maximal ratio $\sup\{h_{l-1}/h_l\colon\ l \geqslant 1\} < \infty$. Usually, the last ratio is constant, e.g. equal to 2. In the following $\gamma = 2$ is fixed (for $\gamma = 1$ compare [6, Corollary 3.8]).

We say that the multi-grid iteration 'converges' if it converges for a suitable choice of h_0 and ν; more precisely if the iteration matrix $M_l = M_l(\nu, h_0, h_1, \ldots, h_l)$ [defined by $u_l^{(i+1)} - u_l = M_l(u_l^{(i)} - u_l)$, $u_l = L_l^{-1} f_l$] satisfies

$$\|M_l\| \leqslant C(\nu) < 1 \quad \text{for } \nu_{\min} \leqslant \nu \leqslant \nu_{\max}(h_1),\ l \geqslant 1, \tag{2.5}$$

where $C(\nu) \longrightarrow 0$ as $\nu \longrightarrow \infty$ and $\nu_{\max}(h) \longrightarrow \infty$ as $h \longrightarrow 0$. The matrix norm $\|\cdot\|$ is associated with some suitable vector norm on V_l.

We recall the following result of [6]. Here and in the sequel C denotes a generic constant independent of l.

PROPOSITION 1. *Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two suitable (not necessarily different) norms on V_l $(l \geqslant 0)$ and define the matrix norms $\|A\|_{i,j}$ $(i, j = 1, 2)$ of A: $V_l \longrightarrow V_m$ by $\sup\{\|Av\|_j/\|v\|_i\colon\ 0 \neq v \in V_l\}$. Assume the smoothing property*

$$\|L_l G_l^\nu\|_{2,1} \leqslant C_0(\nu) h_l^{-\alpha} \quad \textit{for all } l \geqslant 0,\ 1 \leqslant \nu \leqslant \nu_{\max}(h_1) \tag{2.6}$$

with $C_0(\nu) \longrightarrow 0$ $(\nu \longrightarrow \infty)$, $\nu_{\max}(h) \longrightarrow \infty$ $(h \longrightarrow 0)$, and G_l from (2.4) for suitable $\alpha \geqslant 0$. Assume the approximation property

$$\|L_l^{-1} - p_{l,l-1} L_{l-1}^{-1} r_{l-1,l}\|_{1,2} \leqslant C h_{l-1}^\alpha \quad \textit{for all } l \geqslant 1 \tag{2.7}$$

with α from (2.6). Furthermore, the estimates

$$\frac{1}{C}\|v_{l-1}\|_2 \leqslant \|p_{l,l-1} v_{l-1}\|_2 \leqslant C\|v_{l-1}\|_2 \quad \textit{for all } v_{l-1} \in V_{l-1},\ l \geqslant 1, \tag{2.8}$$

$$\|G_l^\nu\|_{2,2} \leqslant C \quad \textit{for all } 1 \leqslant \nu \leqslant \nu_{\max}(h_1),\ l \geqslant 0, \tag{2.9}$$

$$h_l < h_{l-1} \leqslant C h_l \quad \textit{for all } l \geqslant 1 \tag{2.10}$$

are required. Then the multi-grid iteration with $\gamma = 2$ converges: (2.5) holds with $\|\cdot\| = \|\cdot\|_{2,2}$.

3. The Approximation Property.

3.1. *A Criterion Implying the Approximation Property.* Assume

$$r_{l-1,l} L_l p_{l,l-1} = L_{l-1} + \delta_{l-1} \qquad (l \geqslant 1), \tag{3.1a}$$

where δ_{l-1} is small enough in the following sense:

$$\|L_{l-1}^{-1} \delta_{l-1} r'_{l-1,l} L_l^{-1}\|_{1,2} \leqslant C h_{l-1}^\alpha \qquad (l \geqslant 1). \tag{3.1b}$$

$r'_{l-1,l}$ is a suitable restriction involved in (3.3) given below. If L_l is the stiffness matrix of a finite element method, (3.1a) holds with $\delta_{l-1} = 0$ (cf. [6]). δ_{l-1} vanishes, too, if L_{l-1} is defined as in [5].

Moreover, we need the estimate

$$\|L_{l-1}^{-1} r_{l-1,l} L_l\|_{2,2} \leqslant C \qquad (l \geqslant 1) \tag{3.2}$$

and the existence of some linear mapping $r'_{l-1,l}\colon\ V_l \longrightarrow V_{l-1}$ $(l \geqslant 1)$ with

$$\|[I - p_{l,l-1} r'_{l-1,l}] L_l^{-1}\|_{1,2} \leqslant C h_{l-1}^\alpha \quad (l \geqslant 1). \tag{3.3}$$

α involved in (3.1b) and (3.3) is the exponent from (2.6).

LEMMA 1. *Assume that there are norms* $\|\cdot\|_0$ *and* $\|\cdot\|_3$ *on* V_l *such that*

(3.4a) $$\|L_l^{-1}\|_{0,2} \leqslant C, \quad \|L_l^{-1}\|_{1,3} \leqslant C, \quad \|L_l\|_{2,0} \leqslant C \quad (l \geqslant 0).$$

Then (3.1a, b), (3.2), *and* (3.3) *follow from* (3.4b, c, d):

(3.4b) $$\|r_{l-1,l}\|_{0,0} \leqslant C, \quad \|r'_{l-1,l}\|_{3,3} \leqslant C \qquad (l \geqslant 1),$$

(3.4c) $$\|\delta_{l-1}\|_{3,0} \leqslant Ch_{l-1}^{\alpha} \quad (l \geqslant 1),$$

(3.4d) $$\|I - p_{l,l-1}r'_{l-1,l}\|_{3,2} \leqslant Ch_{l-1}^{\alpha} \quad (l \geqslant 1).$$

(3.4d) describes the approximation of grid functions of V_l by $p_{l,l-1}V_{l-1}$: For all $v_l \in V_l$ there is $v_{l-1} \in V_{l-1}$ (namely $r'_{l-1,l}v_l$) with $\|v_l - p_{l,l-1}v_{l-1}\|_2 \leqslant Ch_{l-1}^{\alpha}\|v_l\|_3$. If $\alpha > 0$, $\|\cdot\|_3$ must define a finer topology than $\|\cdot\|_2$.

In Proposition 1 the approximation property (2.7) may be replaced by (3.1)–(3.3):

CRITERION 1. (2.8), (3.1a, b), (3.2), *and* (3.3) *imply the approximation property* (2.7). *By Lemma* 1 *also* (2.8) *and* (3.4a–d) *are sufficient.*

Proof. Since $[I - p_{l,l-1}L_{l-1}^{-1}r_{l-1,l}L_l]\,p_{l,l-1} = -p_{l,l-1}L_{l-1}^{-1}\delta_{l-1}$ by (3.1a), it follows that

$$\begin{aligned}
\|L_l^{-1} - p_{l,l-1}L_{l-1}^{-1}r_{l-1,l}\|_{1,2} &= \|[I - p_{l,l-1}L_{l-1}^{-1}r_{l-1,l}L_l]L_l^{-1}\|_{1,2} \\
&= \|[I - p_{l,l-1}L_{l-1}^{-1}r_{l-1,l}L_l]\,[I - p_{l,l-1}r'_{l-1,l}]L_l^{-1} \\
&\qquad - p_{l,l-1}L_{l-1}^{-1}\delta_{l-1}r'_{l-1,l}L_l^{-1}\|_{1,2} \\
&\leqslant \{1 + \|p_{l,l-1}\|_{2,2}\|L_{l-1}^{-1}r_{l-1,l}L_l\|_{2,2}\}\,\|[I - p_{l,l-1}r'_{l-1,l}]L_l^{-1}\|_{1,2} \\
&\quad + \|p_{l,l-1}\|_{2,2}\|L_{l-1}^{-1}\delta_{l-1}r'_{l-1,l}L_l^{-1}\|_{1,2}.
\end{aligned}$$

Hence, (2.8), (3.1b), and (3.3) yield (2.7). □

Using $r_{l-1,l}[I - L_l p_{l,l-1}L_{l-1}^{-1}r_{l-1,l}] = -\delta_{l-1}L_{l-1}^{-1}r_{l-1,l}$, we obtain a similar result:

CRITERION 2. *Assume* (3.1a),

(2.8*) $$\|r_{l-1,l}\|_{1,1} \leqslant C \qquad (l \geqslant 1),$$

(3.1b*) $$\|L_l^{-1}p'_{l,l-1}\delta_{l-1}L_{l-1}^{-1}\|_{1,2} \leqslant Ch_{l-1}^{\alpha} \qquad (l \geqslant 1),$$

(3.2*) $$\|L_l p_{l,l-1}L_{l-1}^{-1}\|_{1,1} \leqslant C \qquad (l \geqslant 1),$$

(3.3*) $$\|L_l^{-1}(I - p'_{l,l-1}r_{l-1,l})\|_{1,2} \leqslant Ch_{l-1}^{\alpha} \qquad (l \geqslant 1),$$

for a suitable linear mapping $p'_{l,l-1}$: $V_{l-1} \longrightarrow V_l$. *Then* (2.7) *follows.*

3.2. *Application of the Criterion.* In the following we verify the conditions of Criterion 1 for the following example.

Example. Let L_l $(l \geqslant 0)$ be an elliptic difference operator of order $2m$, i.e. the discretization of an elliptic differential operator of order $2m$. Let H_0^s be the space of all complex-valued grid functions defined on the d-dimensional grid $\Omega(h_l) = \{x \in \Omega \subset \mathbf{R}^d\colon\ x/h_l \in \mathbf{Z}^d\}$ endowed with the norm

$$|u|_s = (h/2\pi)^{d/2} \left\| \left[1 + h^{-2} \sum_{j=1}^{d} \sin^2(\xi_j/2)\right]^{s/2} \sum_{x \in \Omega(h)} u(x) e^{ix\xi/h} \right\|_{L^2([-\pi,\pi]^d)} \qquad (s \geq 0),$$

$$|u|_{-s} = \sup\left\{ (h/2\pi)^d \left| \sum_{x \in \Omega(h)} u(x)\bar{v}(x) \right| \Big/ |v|_s\colon\ 0 \neq v \in H_0^s \right\} \qquad (s \geq 0,\ h = h_l)$$

(denoted by $|\cdot|_{s,0}$ in [7]) corresponding to the norm of the Sobolev space $H_0^s(\Omega)$ if $s + 1/2 \neq$ integer. We define the associated matrix norms by

$$|A|_{s,t} = \sup\{|Au|_t/|u|_s\colon\ 0 \neq u \in H_0^s\}.$$

Choose α, $\|\cdot\|_1$, and $\|\cdot\|_2$ (and $\|\cdot\|_0$, $\|\cdot\|_3$ of Lemma 1) by

$$\begin{aligned} &\alpha = \theta + \theta', \quad \|\cdot\|_1 = |\cdot|_{\theta - m}, \quad \|\cdot\|_2 = |\cdot|_{m - \theta'}, \\ &\|\cdot\|_0 = |\cdot|_{-m-\theta'}, \qquad \|\cdot\|_3 = |\cdot|_{m+\theta} \end{aligned} \tag{3.5}$$

for some $\theta, \theta' \in [0, m]$ with $\alpha = \theta + \theta' > 0$. The condition $\alpha > 0$ will be important in Section 4.

The estimate

$$|L_l|_{\vartheta + m,\, \vartheta - m} \leq C(\vartheta) \qquad (l \geq 0,\ \vartheta \in \mathbf{R}) \tag{3.6a}$$

holds if the coefficients of the difference scheme L_l are sufficiently smooth (cf. Lemma 7). In [7] we proved

$$\begin{aligned} &|L_l^{-1}|_{\vartheta - m, \vartheta + m} \leq C \\ &\qquad \text{for all } l \geq 0,\ \vartheta \in [-\theta_0', \theta_0]\ (\theta_0, \theta_0' \in [0, ½),\ \theta_0 + \theta_0' > 0) \end{aligned} \tag{3.6b}$$

under very weak assumptions. The main requirements are stability of L_l with respect to $l_2 = H_0^0$ and ellipticity of L_l. It suffices that the underlying region Ω is Lipschitzian. The assumption on the smoothness of the coefficients is very weak, too. (3.6b) holds even for some schemes with irregular discretizations near the boundary. Symmetry of positive definiteness of L_l are *not* required.

At first we discuss the estimates (3.4a–d) of Lemma 1.

Note 1. (3.5) *and* (3.6a, b) *imply the estimates* (3.4a) *of Lemma* 1 *if* $0 \leq \theta \leq \theta_0$ *and* $0 \leq \theta' \leq \theta_0'$.

For the discussion of (3.4b, d) we restrict our considerations to the case of $m = 1$. Let $h_{l-1}/h_l \in \mathbf{Z}$ and define $p_{l,l-1}^0$ by

$$(p_{l,l-1}^0 u)(x) = \prod_{j=1}^{d} \max\{0,\ 1 - |x_j - y_j|/h_{l-1}\} \qquad (x \in \Omega(h_l),\ y \in \Omega(h_{l-1})),$$

where $u \in V_{l-1}$ is the unit vector with $u(y) = 1$, $u(z) = 0$ for $z \neq y$. $p_{l,l-1}^0$ is an example of an interpolation of order 2. Furthermore, define $r_{l-1,l}^0$ as the mapping adjoint to $p_{l,l-1}^0$: $(u, p_{l,l-1}^0 v) = (r_{l-1,l}^0 u, v)$, where $(v, w) = h^d \Sigma_{x \in \Omega(h)} v(x)\bar{w}(x)$ with $h = h_l$ or h_{l-1}, respectively. In the usual case of $d = 2$ and $h_{l-1} = 2h_l$, the mappings

$p^0_{l,l-1}$ and $r^0_{l-1,l}$ become

$$(p^0_{l,l-1}u)(x) = \begin{cases} u(x) & \text{if } x \in \Omega(h_{l-1}), \\ \frac{1}{2}[u(x+e_jh_l) + u(x-e_jh_l)] & \text{if } x + e_jh_l \in \Omega(h_{l-1}) \\ & \text{or } x - e_jh_l \in \Omega(h_{l-1}), \\ \frac{1}{4}\sum_{j,k=1,2} u(x+(-1)^k e_1h_l + (-1)^j e_2h_l) & \text{otherwise}, \end{cases}$$

$$(r^0_{l-1,l}u)(x) = \frac{1}{4}u(x) + \frac{1}{8}\sum_{j,k=1,2} u(x+(-1)^k e_jh_l)$$

$$+ \frac{1}{16}\sum_{j,k=1,2} u(x+(-1)^j e_1h_l + (-1)^k e_2h_l),$$

where e_j $(j = 1, 2)$ are the unit vectors $(1, 0)$, $(0, 1)$. Note that $u(y) = 0$ if $y \notin \Omega(h_l)$.

Note 2. *Let* $m = 1$. $p^0_{l,l-1}$ *and* $r^0_{l-1,l}$ *defined above satisfy* (3.7a, b):

(3.7a) $\qquad |p^0_{l,l-1}|_{s,s} \leqslant C, \quad |r^0_{l-1,l}|_{s,s} \leqslant C \quad$ *for all* $l \geqslant 1$, $|s| \leqslant 2$,

(3.7b) $\quad |I - p^0_{l,l-1}r^0_{l-1,l}|_{s,t} \leqslant Ch^{s-t}_{l-1} \quad$ *for all* $l \geqslant 1$, $-2 \leqslant t \leqslant s \leqslant 2$, $s - t \leqslant 2$.

COROLLARY 1 TO NOTE 2. *Assume* $m = 1$ *and* (3.5) *and set* $r'_{l-1,l} = r^0_{l-1,l}$. *Then* (2.8) *and the estimates* (3.4b, d) *of Lemma* 1 *hold for* $p_{l,l-1} = p^0_{l,l-1}$ *with* $\alpha = \theta + \theta' > 0$. *Moreover*, (2.8) *and* (3.4d) *remain valid if the coefficients of* $p_{l,l-1}$ *and* $p^0_{l,l-1}$ *differ by* $O(h_l^{1+\theta})$ *and/or if the coefficients of* $p_{l,l-1}$ *and* $p^0_{l,l-1}$ *differ by* $O(1)$ *at points near the boundary (i.e., distance*$(x, \partial\Omega) \leqslant Ch_l$). *Similarly*, (3.4b) *remains true if the coefficients of* $r_{l-1,l}$ *and* $r^0_{l-1,l}$ *differ by* $O(h^{1+\max(\theta,\theta')})$ *or by* $O(1)$ *near the boundary.*

Example. $p_{l,l-1}$ and $r_{l-1,l}$ defined in [5, Eq. (3.4)] satisfy (3.7a, b).

COROLLARY 2 TO NOTE 2. *Generalizations to* $m > 1$ *are obvious.* $p^0_{l,l-1}$ *must be defined by interpolation of order* $> m$.

Note that this requirement is weaker than the requirement "order of interpolation $\geqslant 2m$" of Brandt [3, p. 377].

Since L_l and L_{l-1} should be consistent discretizations of the same differential operator (2.2), the difference $\delta_{l-1} = r_{l-1,l}L_lp_{l,l-1} - L_{l-1}$ is expected to consist of terms of the following form:

$$\delta_{l-1} = \sum_{\beta,\beta' \in Z^d, |\beta|+|\beta'| \leqslant 2m+1} \ \sum_{\gamma \in Z^d} T^\gamma \partial^\beta d_{\gamma,\beta,\beta',l-1}(x, h)\partial^{\beta'},$$

$$\sup\{|d_{\gamma,\beta,\beta',l-1}(x, h)|: \ x \in \Omega(h), l \geqslant 1\} \leqslant Ch^\vartheta,$$

$$\vartheta = \begin{cases} 1 & \text{if } |\beta| + |\beta'| = 2m+1, |\beta|, |\beta'| \leqslant m+1, \\ \theta + \theta' & \text{if } |\beta| + |\beta'| \leqslant 2m, |\beta|, |\beta'| \leqslant m, \end{cases}$$

where β, β', and γ are multi-indices with $|\beta| = \beta_1 + \cdots + \beta_d$, $\partial^\beta = \partial_1^{\beta_1} \cdots \partial_d^{\beta_d}$, $(\partial_j u)(x) = [u(x) - u(x - e_j h]/h$, $(T^\gamma u)(x) = u(x + \gamma h)$, $h = h_{l-1}$ (cf. [7]). The definition of ∂ and T makes sense since $u(x)$ is extended by zero outside $\Omega(h)$. (3.8b) may be replaced by other conditions involving Hölder continuity of $d_{\gamma,\beta,\beta',l-1}(\cdot, h)$.

Example. Consider the differential operator $L = -(a(x_1)u_{x_1})_{x_1} - (b(x_2)u_{x_2})_{x_2}$. Discretize $(au_{x_1})_{x_1}$ by $L_l^{\mathrm{I}}u = -h^{-2}[-a^+u^+ + (a^+ + a^-)u - a^-u^-]$ with $u = u(x)$, $u^\pm = u(x_1 \pm h, x_2)$, $a^\pm = a(x_1 \pm h/2)$. Similarly, L_l^{II} is the discretization of the second term of L. L_l is the sum $L_l^{\mathrm{I}} + L_l^{\mathrm{II}}$ with $h = h_l$. Let $p_{l,l-1} = p_{l,l-1}^0$ and $r_{l-1,l} = r_{l-1,l}^0$ as defined above. Then $\delta_{l-1}^{\mathrm{I}} = r_{l-1,l}L_l^{\mathrm{I}}p_{l,l-1} - L_{l-1}^{\mathrm{I}}$ becomes

$$T_1\partial_1[a(x_1 - h/2) - \tfrac{1}{2}a(x_1 - 3h/4) - \tfrac{1}{2}a(x_1 - h/4)]\,\partial_1$$
$$+ T_1T_2\partial_2^2[-h^2(a(x_1 - h/4) + a(x_1 - 3h/4))/16]\,\partial_1^2$$
$$+ T_2\partial_2^2[-h(a(x_1 + 3h/4) + a(x_1 + h/4) - a(x_1 - h/4) - a(x_1 - 3h/4))/16]\,\partial_1,$$

where $(T_1u)(x) = u(x_1 + h, x_2)$, $(T_2u)(x) = u(x_1, x_2 + h)$ and $h = h_{l-1}$. The brackets contain the coefficients of (3.8a). Obviously, (3.8b) holds if $a(\cdot)$ is Hölder continuous with exponent $\theta + \theta' = \alpha < 1$. If $a(\cdot)$ has Lipschitz continuous derivatives, (3.8b) holds with $\theta = \theta' = 1$.

Note 3. *Let α and the norms be chosen according to* (3.5). (3.8a, b) *implies the estimate* (3.4c) *of Lemma* 1. (3.4c) *holds even if δ_{l-1} contains a further term of order $O(h_{l-1}^{-2m})$ at points near the boundary.*

Proof. Use $|\partial^\beta u|_0 \leqslant C|u|_{m+\theta}$ if $|\beta| \leqslant m + \theta$ and $h|\partial^\beta u|_0 \leqslant Ch^\theta|u|_{m+\theta}$ if $m + \theta \leqslant |\beta| = m + 1$. For perturbations near the boundary apply the following lemma (cf. [7]). □

LEMMA 2. *Let $\Omega(h)$ have 'property* C' *defined in* [7]. *Assume that the subset $\Gamma(h) \subset \Omega(h)$ satisfy distance $(x, h\mathbf{Z}^d\backslash\Omega(h)) \leqslant Ch$ for some $C \neq C(h)$ and all $x \in \Gamma(h)$, that means, all points of $\Gamma(h)$ have a distance less than Ch from the boundary. Define the restriction γ by $(\gamma u)(x) = u(x)$ if $x \in \Gamma(h)$, $(\gamma u)(x) = 0$ otherwise. Then $|\gamma|_{s,t} \leqslant C'h^{s-t}$ is valid.*

A sufficient condition for 'property C' is that Ω is Lipschitzian.

From Notes 1–3, Lemma 1 and Criterion 1 one concludes that the approximation property (2.7) holds for a very general class of difference schemes L_l.

Example (*Application to the Shortley-Weller scheme*). Discretize $-\Delta u = f$ (in a Lipschitz region $\Omega \subset \mathbf{R}^2$), $u = g$ (on $\partial\Omega$) by the Shortley-Weller scheme L_l (cf. [5], [8, p. 203ff.]). In [7, Note 2.3] we proved (3.6b) with $\theta_0' = 0$, $\theta_0 > 0$. But note that (3.6a) is not valid since the diagonal D_l of the matrix L_l can be arbitrarily large. Nevertheless, $(h_l^2D_l)^{-1}L_l$ and $L_l(h_l^2D_l)^{-1}$: $H_0^{\theta+m} \longrightarrow H_0^{\theta-m}$ are uniformly bounded.

Define $\theta' = 0$, $p_{l,l-1} = p_{l,l-1}^0$, $r_{l-1,l} = r_{l-1,l}^0(h_l^2D_l)^{-1}$, $r_{l-1,l}' = (h_{l-1}^2D_{l-1})^{-1}r_{l-1,l}^0$ (or define $p_{l,l-1}$ and $r_{l-1,l}$ as in [5]). Then (3.1a, b), (3.2), and (3.3) are fulfilled. For a proof modify Lemma 1: Split $r_{l-1,l}L_l$ into $r_{l-1,l}^0 \cdot [(h_l^2D_l)^{-1}L_l]$ and $\delta_{l-1}r_{l-1}'$, into $[\delta_{l-1}(h_{l-1}^2D_{l-1})^{-1}] \cdot r_{l-1,l}^0$. Thus, we have shown the approximation property (2.7) with $\alpha = \theta > 0$ by means of Criterion 1.

4. Criteria Implying the Smoothing Property.

4.1. *Preparing Lemmata.* The following lemma describes a norm equivalent to $|\cdot|_s$.

LEMMA 3. *Let $\Omega(h)$ have 'property C' (cf. Lemma 2). Assume $L_{l,0}$ to be a positive definite and H_0^m-elliptic difference operator of order $2m$, i.e., $L_{l,0} = L_{l,0}^*$ and $|u|_m^2/C \leqslant (L_{l,0}u, u) \leqslant C|u|_m^2$, where $(u, v) = h^d \Sigma_{x\in\Omega(h)} u(x)\bar{v}(x)$. The fractional powers of $\Lambda := (L_{l,0})^{1/(2m)}$ are well defined. Then $|u|_s$ and $|\Lambda^s u|_0$ are equivalent: $(1/C')|u|_s \leqslant |\Lambda^s u|_0 \leqslant C'|u|_s$, for $-m \leqslant s \leqslant m$. C' does not depend on h_l.*

Proof. Use Lemma 2.1 of [7] and the following lemma. □

LEMMA 4 (INTERPOLATION). *Let H_1 and H_2 be two Hilbert spaces. $A\colon H_1 \longrightarrow H_2$, $\Lambda_i\colon H_i \longrightarrow H_i$ and $\Lambda_i^{-1}\colon H_i \longrightarrow H_i$ $(i = 1, 2)$ are assumed to be bounded. Furthermore, let Λ_1 and Λ_2 be positive definite. Then the inequality*

$$\|\Lambda_2^\gamma A \Lambda_1^{-\gamma}\|_{H_1\to H_2} \leqslant C_1^{(\gamma_2-\gamma)/(\gamma_2-\gamma_1)} C_2^{(\gamma-\gamma_1)/(\gamma_2-\gamma_1)}$$

holds for all $\gamma \in [\gamma_1, \gamma_2]$ if it is valid for $\gamma = \gamma_1$ and $\gamma = \gamma_2$.

Proof. Set $\varphi(\gamma) = \|\Lambda_2^\gamma A \Lambda_1^{-\gamma}\|_{H_1\to H_2}$ and note that

$$\varphi(\gamma)^2 = \|\Lambda_2 A \Lambda_1^{-2\gamma} A^* \Lambda_2^\gamma\|_{H_2\to H_2} = \rho(\Lambda_2^{\gamma'} A \Lambda_1^{-\gamma'} \Lambda_1^{-\gamma''} A^* \Lambda_2^{\gamma''}) \leqslant \varphi(\gamma')\varphi(\gamma'')$$

for all γ', γ'' with $\gamma' + \gamma'' = \gamma$ (ρ: spectral radius). Therefore, the estimate follows by bisection for all $\gamma = \gamma_1 + \nu 2^{-\mu}(\gamma_2 - \gamma_1)$ with $\nu, \mu \in \mathbf{Z}$, $\mu \geqslant 0$, $0 \leqslant \nu \leqslant 2^\mu$. The continuity of $\varphi(\gamma)$ concludes the proof. □

The preceding lemmata yield the following estimates.

LEMMA 5. *The estimates* (4, 1a, b, c) *hold with C independent of h_l:*

$$|A|_{r,r} \leqslant C|A|_{s,s}^{(t-r)/(t-s)}|A|_{t,t}^{(r-s)/(t-s)} \qquad (-m \leqslant s \leqslant r \leqslant t \leqslant m), \tag{4.1a}$$

$$|A|_{r,-r} \leqslant C|A|_{0,0}^{(t-r)/t}|A|_{t,-t}^{r/t} \qquad (0 \leqslant r \leqslant t \leqslant m \text{ or } 0 \geqslant r \geqslant t \geqslant -m), \tag{4.1b}$$

$$|A|_{r,-s} \leqslant C|A|_{0,0}^{(2m-r-s)/(2m)}|A|_{2m,0}^{r/(2m)}|A|_{0,-2m}^{s/(2m)} \qquad (r \geqslant 0, s \geqslant 0, r + s \leqslant 2m). \tag{4.1c}$$

Proof. By virtue of Lemma 3, $|u|_s$ can be replaced with $|\Lambda^s u|_0$. Hence, $|A|_{r,s}$ becomes $|\Lambda^s A \Lambda^{-r}|_{0,0}$. Applying Lemma 4 with $\Lambda_1 = \Lambda_2 = \Lambda$ we obtain (4.1a). (4.1b) follows by choosing $\Lambda_1 = \Lambda$, $\Lambda_2 = \Lambda^{-1}$. For the proof of (4.1c) apply Lemma 3 and (4.1a) with $2m$ instead of m. We abbreviate $|A|_{p,q}$ by $a(p, q)$. Lemma 4 with $\Lambda_1 = I$, $\Lambda_2 = \Lambda$ yields $a(r + s, 0) \leqslant Ca(0, 0)^{1-\beta}a(2m, 0)^\beta$ with $\beta = (r + s)/(2m)$. Similarly, $a(0, -r - s) \leqslant Ca(0, 0)^{1-\beta}a(0, -2m)^\beta$ follows. Applying (4.1a) to $\Lambda^{-r-s}A$ instead of A, one obtains $a(r, -s) \leqslant Ca(r + s, 0)^{r/(r+s)}a(0, -r - s)^{s/(r+s)}$. Inserting the estimates of $a(r + s, 0)$ and $a(0, -r - s)$ we are led to (4.1c). □

Smoothing by Gauss-Seidel's iteration is expressed by

$$G_l(v_l, f_l) = (D_l - R_l)^{-1}(S_l v_l + f_l), \qquad G_l = (D_l - R_l)^{-1}S_l, \tag{4.2a}$$

where

(4.2b) $$L_l = D_l - R_l - S_l.$$

Definition 1. *The splitting* (4.2b) *is called* 2*-cyclic* (*cf.* [10, *p.* 39]) *if there are two distinct subsets* $\Omega_1(h)$ *and* $\Omega_2(h)$ *of* $\Omega(h)$ *with* $\Omega_1(h) \cup \Omega_2(h) = \Omega(h)$ *such that*

$$D_l = \omega_1 L_l \omega_1 + \omega_2 L_l \omega_2, \quad R_l = -\omega_2 L_l \omega_1, \quad S_l = -\omega_1 L_l \omega_2,$$

where the restrictions ω_j *are defined by* $(\omega_j u)(x) = u(x)$ *if* $x \in \Omega_j(h)$ *and* $(\omega_j u)(x) = 0$ *otherwise.*

Throughout this section we shall assume

(4.3) $\Omega(h)$ have 'property C' (cf. Lemma 2); $\alpha, \|\cdot\|_1, \|\cdot\|_2$ be defined by (3.5).

LEMMA 6. *Let the splitting* (4.2b) *be* 2*-cyclic and assume* $L_l = L_l^*$ *to be positive definite. Then* $|L_l G_l^\nu|_{0,0} \leqslant |D_l|_{0,0}/(\nu + 1/2)$ *holds for all* $\nu \geqslant 1$.

Proof. Numbering first the grid points of $\Omega_1(h)$ yields the following block structure:

$$L_l = \begin{bmatrix} d_1 & -s \\ -r & d_2 \end{bmatrix}, \quad G_l = \begin{bmatrix} 0 & d_1^{-1}s \\ 0 & d_2^{-1}rd_1^{-1}s \end{bmatrix},$$

$$L_l G_l^\nu = \begin{bmatrix} 0 & s\{[d_2^{-1}rd_1^{-1}s]^{\nu-1} - [d_2^{-1}rd_1^{-1}s]^\nu\} \\ 0 & 0 \end{bmatrix}.$$

Hence, $|D_l^{-1/2} L_l G_l^\nu D_l^{-1/2}|_{0,0}^2 = |[\begin{smallmatrix} 0 & 0 \\ 0 & A \end{smallmatrix}]|_{0,0}$ follows from $s^* = r$ with $A = B^{2\nu-1}(I - B)^2$ and $B = B^* = d_2^{-1/2} r d_1^{-1} s d_2^{-1/2}$. It is well known that $\rho(G_l) = \rho(B) = \|B\| \leqslant 1$ (cf. Note 5), where $\|\cdot\|$ denotes the $|\cdot|_{0,0}$-norm restricted to the last block.

$$\begin{aligned} \|A\| = \rho(A) &= \sup\{|\lambda^{2\nu-1}(1-\lambda)^2|: \ \lambda \in \text{spectrum of } B\} \\ &\leqslant \sup\{\lambda^{2\nu-1}(1-\lambda)^2: \ 0 \leqslant \lambda \leqslant 1\} \leqslant 1/(\nu + 1/2)^2 \end{aligned}$$

implies $|L_l G_l^\nu|_{0,0} \leqslant |D_l^{1/2}|_{0,0}^2 \|A\|^{1/2} \leqslant |D_l|_{0,0}/(\nu + 1/2)$. $\square$

4.2. *General Criteria.*

CRITERION 3. *Assume* (4.3), $\theta = \theta'$, (3.6a) *for* $\vartheta = 0$, *and*

(4.4a) $$|L_l G_l^\nu|_{0,0} \leqslant h_l^{-2m} C(\nu) \quad \textit{for } 1 \leqslant \nu \leqslant \nu_{\max}(h_1),\ l \geqslant 0;$$
$$C(\nu) \longrightarrow 0\ (\nu \longrightarrow \infty), \qquad \nu_{\max}(h) \longrightarrow \infty\ (h \longrightarrow 0),$$

(4.4b) $$|G_l^\nu|_{m,m} \leqslant C \quad \textit{for all } 1 \leqslant \nu \leqslant \nu_{\max}(h_1),\ l \geqslant 0.$$

Then the smoothing property (2.6) *holds with* $C_0(\nu) = C'[C(\nu)]^{\theta/m}$.

Proof. (3.6a) ($\vartheta = 0$) and (4.4b) yield $|L_l G_l^\nu|_{m,-m} \leqslant |L_l|_{m,-m} |G_l^\nu|_{m,m} \leqslant C$. Hence, (4.1b) ($t = m$, $r = m - \theta$) implies (2.6). $\square$

The following criterion applies also to the case of $\theta \neq \theta'$:

CRITERION 3*. *Assume* (4.3), (4.4a) *and*

$$|L_l|_{2m,0} \leqslant C, \quad |L_l^*|_{2m,0} \leqslant C, \quad |G_l^\nu|_{0,0} \leqslant C,$$

(4.4b*)
$$|\widetilde{G}_l^\nu|_{0,0} \leqslant C \quad (0 \leqslant \nu \leqslant \nu_{\max}(h_1), l \geqslant 0),$$

where $\widetilde{G}_l = L_l G_l L_l^{-1}$. *Then the smoothing property* (2.6) *holds with* $C_0(\nu) = C' \cdot C(\nu)^{\alpha/(2m)}$ $[\alpha = \theta + \theta'$, *cf.* (3.5)$]$.

Proof. (4.4b*) implies $|L_l G_l^\nu|_{2m,0} = |\widetilde{G}_l^\nu L_l|_{2m,0} \leqslant |\widetilde{G}_l^\nu|_{0,0} |L_l|_{2m,0} \leqslant C$. Since $|L_l^*|_{2m,0} = |L_l|_{0,-2m}$, also $|L_l G_l^\nu|_{0,-2m} \leqslant |L_l|_{0,-2m} |G_l^\nu|_{0,0} \leqslant C$ holds. (4.1c) yields (2.6). □

First we shall verify the conditions of Criteria 3 and 3* for positive definite schemes. In a second step it is shown that additional terms of lower order may be added. Hence, all difference schemes with a hermitian principle part satisfy the smoothing property. In a third step we treat perturbations of order $O(h_l^{-2m})$ located at points near the boundary. Such perturbations often arise from special discretizations at the boundary.

Usually, the function $C_0(\nu)$ of (2.6) is $C/(\nu + 1)^{\alpha/(2m)}$. Therefore, $C_0(\nu) \longrightarrow 0$ requires $\alpha = \theta + \theta' > 0$. The choice of $\theta = \theta' = 0$ is excluded. The upper bound $\nu_{\max}(h)$ of ν in (2.6) may be omitted (i.e. $\nu_{\max} = \infty$) if L_l is positive definite. In the case of other schemes $\nu_{\max}(h)$ might become finite (but $\nu_{\max}(h) \longrightarrow \infty$ as $h \longrightarrow 0$).

4.3. *Case of Positive Definite Difference Schemes.* Throughout this subsection we assume

(4.5)
$$L_l = L_l^*, \quad \frac{1}{C} |u|_m \leqslant (L_l u, u) \leqslant C |u|_m$$

as in Lemma 3. The proofs of convergence in [1], [2], [4], [6], [9], [11] require smoothing by

(4.6)
$$G_l(v_l, f_l) = v_l - \omega_l h_l^{2m}(L_l v_l - f_l), \quad G_l = I - \omega_l h_l^{2m} L_l.$$

If the diagonal of L_l is a multiple of I, G_l corresponds to a damped Jacobi iteration.

Note 4 (*Smoothing by Jacobi Iteration*). *Assume* (4.3), (4.5), (4.6) *and* $0 \leqslant \omega_l \leqslant h_l^{-2m}/|L_l|_{0,0}$. *Then the smoothing property* (2.6) *holds for all* ν ($\nu_{\max} = \infty$) *with*

(4.7)
$$C_0(\nu) = C/(\nu + ½)^{\alpha/(2m)} \qquad (\alpha = \theta + \theta' \textit{ from } (3.5)).$$

Proof. One may choose $L_{l,0} = L_l$ in Lemma 3. Thus, it suffices to estimate $A = \Lambda^{\theta - m} L_l G_l^\nu \Lambda^{\theta' - m} = L_l^\beta (I - \omega_l h_l^{2m} L_l)^\nu$, $\beta = \alpha/(2m)$, with respect to $|\cdot|_{0,0}$. But this norm is equal to the spectral radius. Since the spectrum of L_l is contained in $[0, 1/(\omega_l h_l^{2m})]$,

$$\rho(A) = \sup\{\lambda^\beta (1 - \omega_l h_l^{2m} \lambda)^\nu : \; 0 \leqslant \omega_l h_l^{2m} \leqslant 1\} \leqslant C/(\nu + 1)^\beta$$

proves Note 4. □

The techniques of the following subsections can be applied to smoothing by (4.6), too. But since we are mainly interested in smoothing by Gauss-Seidel's iteration, henceforward our considerations are restricted to this subject.

Note 5 (*Smoothing by Gauss-Seidel*). *Assume* (4.3), $\theta = \theta'$, (4.5). *Let* G_l *be defined by* (4.2a, b), *where the splitting* (4.2b) *is required to be* 2-*cyclic. Then the smoothing property* (2.6) *holds with* $C_0(\nu)$ *from* (4.7) *for all* ν ($\nu_{\max}(h) = \infty$).

Proof. Since (4.2b) is 2-cyclic and L_l is positive definite, D_l is positive definite, too. Thus, the theorem of Ostrowski (cf. [8, p. 297], [10, p. 77]) applies resulting in $|L_l^{1/2} G_l^\nu L_l^{-1/2}|_{0,0} \leqslant 1$ ($\nu \geqslant 0$). By Lemma 3 (4.4b) follows. Lemma 6 implies (4.4a) with $C(\nu) = C/(\nu + 1/2)$ since $|D_l|_{0,0} \leqslant |L_l|_{0,0} \leqslant Ch_l^{-2m}$ results from (4.5) and $|u|_m \leqslant Ch_l^{-m}|u|_0$. (3.6a) with $\vartheta = 0$ holds by virtue of (4.5). Hence, all conditions of Criterion 3 are satisfied. □

Example. Let $L_l u = f$ be the discretization of $-\operatorname{div}[(a(x_1), b(x_2))^T \operatorname{grad} u] = \varphi$ in Ω and $u = 0$ on $\partial\Omega$ as in the example of Section 3.1. (4.5) holds if $a(x_1), b(x_2) \in [\epsilon, C] \subset (0, \infty)$. Use the 'red-black' ordering of the grid points: $\Omega_1(h) = \{x \in \Omega(h)\colon (x_1 + x_2)/h \text{ even}\}$. If, in addition, Ω is a Lipschitz region, all conditions of Note 5 are satisfied. The smoothing property holds for all $\theta = \theta' = \alpha/2 \in (0, m]$.

Note 5 illustrates the application of Criterion 3. In order to apply Criterion 3* the following lemmata give conditions implying (4.4b*).

LEMMA 7. *The inequalities* $|L_l|_{2m,0} \leqslant C$ *and* $|L_l^*|_{2m,0} \leqslant C$ *hold if the coefficients are sufficiently smooth. More precisely, the estimates hold if* L_l *is a finite sum of terms of the form*

$$T^\gamma \partial^\beta c(x, h_l) \partial^{\beta'} \qquad (\gamma, \beta, \beta' \in \mathbf{Z}^d, \beta_j \geqslant 0, \beta_j' \geqslant 0, |\beta| + |\beta'| \leqslant 2m),$$

where all kth derivatives of $c(x, h_l)$ *with respect to* x *are uniformly Lipschitz continuous on* $\bar{\Omega}$ *for* $k = \max(|\beta|, |\beta'|) - 1$ [*for* T^γ *and* ∂^β *compare Section* 3, (3.8a)].

Proof. $|L_l|_{2m,0} \leqslant C$ requires $k \geqslant |\beta| - 1$. Since L_l^* contains $(-1)^{|\beta|+|\beta'|} T^{\beta'} \partial^{\beta'} c \partial^\beta T^{\gamma+\beta}$, also $k \geqslant |\beta'| - 1$ must hold. □

LEMMA 8. *The estimates* $|G_l^\nu|_{0,0} \leqslant C$, $|\tilde{G}_l^\nu|_{0,0} \leqslant C$ *are valid for all* $\nu \geqslant 0$ *and* $l \geqslant 0$ *if the splitting* (4.2b) *is* 2-*cyclic and if one of the following conditions holds*:

(4.8a) $\qquad L_l$ *satisfies* (4.5), $\quad |D_l^{-1}|_{0,0} \leqslant Ch_l^{2m}$,

(4.8b) $\quad D_l = \omega_l h_l^{-2m} I$, $\quad L_l$ *and* L_l^* *are diagonally dominant* (*cf.* [10, *p.* 23]),

(4.8c) $\qquad |D_l^{-1}(R_l + S_l)|_{0,0} \leqslant 1, \quad |(R_l + S_l)D_l^{-1}|_{0,0} \leqslant 1.$

Note that $|\cdot|_{0,0}$ coincides with the usual spectral norm of matrices.

Proof. (a) One verifies that $\tilde{G}_l = S_l(D_l - R_l)^{-1}$. G_l^ν and $\tilde{G}_l^\nu$ have the representations

$$G_l^\nu = \begin{bmatrix} 0 & d_1^{-1}s[d_2^{-1}rd_1^{-1}s]^{\nu-1} \\ 0 & [d_2^{-1}rd_1^{-1}s]^\nu \end{bmatrix}, \quad \tilde{G}_l^\nu = \begin{bmatrix} [sd_2^{-1}rd_1^{-1}]^\nu & [sd_2^{-1}rd_1^{-1}]^{\nu-1}sd_2^{-1} \\ 0 & 0 \end{bmatrix} \qquad (\nu \geqslant 1)$$

Assume (4.8a) and let B be as in the proof of Lemma 6. $|D_l^{1/2} G_l^\nu D_l^{-1/2}|_{0,0}^2 = |D_l^{-1/2} \tilde{G}_l^\nu D_l^{1/2}|_{0,0}^2 = \|B^{2\nu} + B^{2\nu-1}\| \leqslant 2$ shows $|G_l^\nu|_{0,0} \leqslant \sqrt{2}|D_l^{-1/2}|_{0,0}|D_l^{1/2}|_{0,0} \leqslant C$ and $|\tilde{G}_l^\nu|_{0,0} \leqslant C$.

(b) Let $\|\cdot\|_\infty$ be the matrix norm corresponding to the supremum norm. (4.8b) implies that the $\|\cdot\|_\infty$ norm of $D_l^{-1}(R_l + S_l) = (R_l + S_l)D_l^{-1}$ and of the adjoint matrix are bounded by 1. Hence, (4.8c) holds.

(c) From (4.8c) it follows that $\|d_1^{-1}s\|, \|d_2^{-1}r\|, \|sd_2^{-1}\|, \|rd_1^{-1}\| \leq 1$ ($\|\cdot\|$: spectral norm). Then the representations of G_l^ν and $\widetilde{G}_l^\nu$ yield $|G_l^\nu|_{0,0}, |\widetilde{G}_l^\nu|_{0,0} \leq \sqrt{2}$. □

We summarize:

Note 6. *Assume* (4.3) *and* (4.5). *Let the coefficients of* L_l *be sufficiently smooth (cf. Lemma* 7). G_l *is defined by* (4.2a), *where the splitting* (4.2b) *is* 2-*cyclic with* $|D_l^{-1}|_{0,0} \leq Ch_l^{2m}$. *Then the smoothing property* (2.6) *holds for all* $\theta, \theta' \in [0, m]$, $\theta + \theta' = \alpha > 0$ *with* $C_0(\nu)$ *from* (4.7) *and* $\nu_{\max}(h) = \infty$.

Proof. (4.4a) follows as in Note 5. Thanks to Lemmata 7, 8 the Criterion 3* yields (2.6). □

4.4. *Perturbations by Lower Order Terms.* In the following we shall assume that the difference scheme L_l is the sum $L_l' + L_l''$, where L_l' satisfies the smoothing property (2.6). We assume a 2-cyclic splitting of L_l and L_l':

$$L_l = D_l - R_l - S_l, \quad L_l' = D_l' - R_l' - S_l',$$

$$G_l = (D_l - R_l)^{-1}S_l, \quad G_l' = (D_l' - R_l')^{-1}S_l', \tag{4.9}$$

$$G_l'' = G_l - G_l', \quad D_l'' = D_l - D_l', \quad R_l'' = R_l - R_l', \quad S_l'' = S_l - S_l'.$$

L_l'' is called a lower order term if there is some $\beta > 0$ such that

$$|L_l''|_{0,0} \leq Ch_l^{\beta - 2m} \quad (\beta > 0, l \geq 0). \tag{4.10}$$

The first criterion applies if $\beta > m - \max(\theta, \theta')$.

CRITERION 4. *Let* $L_l = L_l' + L_l''$ *and* L_l' *have* 2-*cyclic splittings and define* G_l *and* G_l' *by* (4.9). *Choose the norms by* (3.5) *and assume*

$$|L_l''|_{0,\theta - m} \leq Ch_l^{\beta - m - \theta} \quad [\textit{or } |L_l''|_{m - \theta', 0} \leq Ch_l^{\beta - m - \theta'}], \tag{4.10*}$$

$$|D_l'^{-1}|_{0,0} \leq Ch_l^{2m}, \quad |L_l'|_{m,-m} \leq C, \tag{4.11}$$

and $\beta > m - \theta'$ [*or* $\beta > m - \theta$, *respectively*]. *Then* L_l *has the smoothing property* (2.6) *if* L_l' *has.*

For the usual case of $m = 1$ β takes the values 1 and 2. Hence, $\alpha = \theta + \theta' > 0$ implies $\beta > m - \theta'$ or $\beta > m - \theta$. (4.10*) holds if L_l'' is a difference scheme of order $\leq 2m - \beta$ with smooth coefficients (cf. Lemma 7). Note that (4.10*) implies (4.10).

Proof. By (4.10) and (4.11) the estimate $|D_l'^{-1}L_l''|_{0,0} \leq Ch_l^\beta$ holds. The same norm of $D_l'^{-1}D_l''$, $D_l'^{-1}R_l''$, and $D_l'^{-1}S_l''$ is also of order $O(h_l^\beta)$ since the splitting is 2-cyclic. Hence, $|G_l''|_{0,0} \leq Ch_l^\beta$ is valid for sufficiently small h_l. The second estimate of (4.11) implies $|L_l'|_{0,0} \leq Ch_l^{-2m}$. Thus, $|G_l'|_{0,0} \leq C$ holds, too. $X(\nu) = G_l^\nu - G_l'^\nu$

can be estimated by

$$|X(\nu)|_{m-\theta',0} \leqslant |X(\nu)|_{0,0} \leqslant c(\nu, h_l) := \sum_{\mu=1}^{\nu} \binom{\nu}{\mu} |G_l'|_{0,0}^{\mu} |G_l''|_{0,0}^{\nu-\mu}$$
$$\leqslant \sum_{\mu=1}^{\nu} \binom{\nu}{\mu} C^{\mu} [Ch_l^{\beta}]^{\nu-\mu} = C^{\nu}[(1 + h_l^{\beta})^{\nu} - 1].$$

The further terms of

$$\|L_l G_l^{\nu}\|_{2,1} \leqslant \|L_l' G_l'^{\nu}\|_{2,1} + |L_l''|_{0,\theta-m} |G_l^{\nu}|_{m-\theta',0} + |L_l'|_{0,\theta-m} |X(\nu)|_{m-\theta',0}$$

are bounded by

$$|L_l''|_{0,\theta-m} \leqslant Ch_l^{\beta-m-\theta}, \qquad |L_l'|_{0,\theta-m} \leqslant Ch_l^{-m-\theta}, \qquad |G_l^{\nu}|_{m-\theta',0} \leqslant C^{\nu}.$$

Since L_l' satisfies (2.6) with $C_0'(\nu)$ and $\nu_{\max}'(h)$ one obtains $\|L_l G_l^{\nu}\|_{2,1} \leqslant h_l^{-\alpha} c_0(\nu, h_l)$ with $c_0(\nu, h) = C_0'(\nu) + h^{\theta'-m+\beta} C^{\nu} + Ch^{\theta'-m} c(\nu, h)$. $\theta' - m + \beta > 0$ implies $c_0(\nu, 0) = C_0'(\nu)$. Thus, there exists $\nu_{\max}(h) \leqslant \nu_{\max}'(h)$ with $\nu_{\max}(0) = \infty$ such that $c_0(\nu, h) \leqslant C_0(\nu) := 2C_0'(\nu)$ for all $0 \leqslant \nu \leqslant \nu_{\max}(h)$. In the case of the second inequality of (4.10*) and $\beta - m < \theta$ apply the analogous estimates to $\widetilde{G}_l^{\nu} L_l = L_l G_l^{\nu}$. □

The following criterion is applicable for all $\beta > 0$. On the other hand L_l' must satisfy not only the smoothing property but also the sufficient conditions of Criterion 3*.

CRITERION 5. *Let $L_l = L_l' + L_l''$ and L_l' have 2-cyclic splittings and define G_l and G_l' by* (4.9). *Assume* (4.3), (4.10), (4.11), *and $|L_l''|_{2m,0} \leqslant C$, $|L_l''^*|_{2m,0} \leqslant C$. Moreover, the estimates* (4.4a) *and* (4.4b*) *must be valid for L_l', G_l', $\widetilde{G}_l'$ (instead of L_l, G_l, $\widetilde{G}_l$). Then the smoothing property* (2.6) *holds for L_l, too.*

Proof. Repeating the proof of Criterion 4 for the special case of $\theta = \theta' = m$ one obtains (4.4a). The same proof shows (4.4b*) for a suitable choice of $\nu_{\max}(h)$. Hence Criterion 3* implies (2.6). □

Note 6 and Criterion 5 establish the following result.

Note 7. *Assume* (4.3) *and $C^{-1}|u|_m^2 \leqslant \mathrm{Re}(L_l u, u) + \lambda_0 |u|_0^2 \leqslant C|u|_m^2$ for some real λ_0 (H_0^m-coerciveness of L_l). L_l must consist of the terms $T^{\gamma}\partial^{\beta}c(x, h_l)\partial^{\beta'}$ described in Lemma* 7. *G_l is defined by* (4.2a), *where the splitting* (4.2b) *is 2-cyclic with $|D_l^{-1}|_{0,0} \leqslant Ch_l^{2m}$. Then the smoothing property* (2.6) *holds with $C_0(\nu)$ from* (4.7).

Proof. Define $L_l' = (L_l + L_l^*)/2 + \lambda_0 I$ and $L_l'' = L_l - L_l'$. $|L_l''|_{0,0} \leqslant Ch_l^{2m-1}$ and $|D_l^{-1}|_{0,0} \leqslant Ch_l^{2m}$ imply $|D_l'^{-1}|_{0,0} \leqslant C'h_l^{2m}$ for sufficiently small h_l. Hence, Note 6 shows that (4.4a) and (4.4b*) hold for L_l' and G_l'. (2.6) follows by Criterion 5. □

4.5. *Perturbation at the Boundary.* In particular, if special discretizations are used at points near the boundary, the difference scheme L_l is a sum of a scheme L_l' with smooth coefficients as studied in the foregoing section and a further term L_l'' with $(L_l''u)(x) \neq 0$ only at points near the boundary. The following note shows the smoothing property for an important class of discretizations.

Note 8. *Let $L_l = L_l' + L_l''$ and L_l' have 2-cyclic splittings with diagonal matrices D_l, D_l' and define G_l and G_l' by* (4.9). *If $(L_l''u)(x) \neq 0$ for some u, $|x - x'| \leqslant Ch_l$ must hold for some $x' = \nu' h_l \notin \Omega(h_l)$ (cf. Lemma 2). Moreover, $(L_l u)(x)$ and*

$(L_l'u)(x)$ must depend only on $u(x')$ with $|x'-x| \leqslant Ch_l (x, x' \in \Omega(h_l))$. Assume that (4.4a) *and* (4.4b*) *hold for L_l', G_l', and $\widetilde{G}_l'$ with $C(\nu)$ from* (4.7) *(sufficient conditions are those of Note* 7). *Furthermore,* (4.3) *and* (3.6b) *with some $\vartheta \in (0, 1/2)$ are required for L_l' (instead of L_l). Let*

$$\text{(4.12a)} \qquad \begin{gathered} |D_l'^{-1}|_{0,0} \leqslant Ch_l^{2m}, \quad |R_l + S_l|_{0,0} \leqslant Ch_l^{-2m}, \\ |D_l'^{-1}L_l'|_{0,0} \leqslant C, \quad |L_l'D_l'^{-1}|_{0,0} \leqslant C. \end{gathered}$$

The inequalities

$$\text{(4.12b)} \quad 0 \leqslant D_l^{-1}(R_l + S_l) \leqslant D_l'^{-1}(R_l' + S_l'), \qquad 0 \leqslant (R_l + S_l)D_l^{-1} \leqslant (R_l' + S_l')D_l'^{-1}$$

must hold for all entries of the matrices. Then the smoothing property (2.6) *is valid with the same $\nu_{\max}(h)$ as for L_l'.*

It is to be emphasized that D_l is not required to be uniformly bounded.

Proof. (1) We abbreviate $|\cdot|_{0,0}$ by $\|\cdot\|$. There is γ as in Lemma 2 such that $L_l'' = L_l''\gamma$. (4.4b*) implies $\|D_l'\| \leqslant Ch_l^{-2m}$ and $\|L_l'\| \leqslant Ch_l^{-2m}$. By virtue of the Perron-Frobenius theory (cf. [10, p. 26]) $\|D_l^{-1}L_l\| \leqslant \|D_l'^{-1}L_l'\| \leqslant C$ can be concluded from (4.12a, b). Therefore,

$$\begin{aligned} |D_l'D_l^{-1}L_l|_{2m,0} &\leqslant |D_l'D_l^{-1}L_l - L_l'|_{2m,0} + |L_l'|_{2m,0} \leqslant |(D_l'D_l^{-1}L_l - L_l')\gamma|_{2m,0} + C \\ &\leqslant (\|D_l'\| \, \|D_l^{-1}L_l\| + \|L_l'\|)|\gamma|_{2m,0} + C \leqslant C' \end{aligned}$$

yields the first inequality of (4.13a):

$$\text{(4.13a)} \qquad |D_l'D_l^{-1}L_l|_{2m,0} \leqslant C, \qquad |D_l'^*D_l^{*-1}L_l^*|_{2m,0} \leqslant C.$$

Similarly, the second estimate is proved.

(2) Let d_1, d_2, r, and s be as in the proof of Lemma 6. (4.12b) yields $0 \leqslant d_1^{-1}s \leqslant d_1'^{-1}s'$, etc. Hence

$$\text{(4.13b)} \qquad 0 \leqslant G_l^\nu \leqslant G_l'^\nu, \quad 0 \leqslant \widetilde{G}_l^\nu \leqslant \widetilde{G}_l'^\nu \qquad (\nu \geqslant 0)$$

follows. The Perron-Frobenius theory shows $\|G_l^\nu\| \leqslant \|G_l'^\nu\|$. By $\|D_lG_l^\nu\| \leqslant \|s\|(1 + \|rd_1^{-1}\|)\|G_l^{\nu-1}\| \leqslant Ch_l^{-2m}$ [cf. (4.12a), (4.4b*)] and $\|D_l'^{-1}\| \leqslant Ch_l^{2m}$ we obtain the first estimate of (4.13c):

$$\text{(4.13c)} \qquad \|D_l'^{-1}D_lG_l^\nu\| \leqslant C, \quad \|\widetilde{G}_l^\nu D_l D_l'^{-1}\| \leqslant C \qquad (1 \leqslant \nu \leqslant \nu_{\max}(h_1)).$$

The proof of the second one is similar.

(3) (4.13a) and (4.13b) imply

$$\text{(4.13d)} \qquad |L_lG_l^\nu|_{0,-2m} \leqslant C, \quad |L_lG_l^\nu|_{2m,0} \leqslant C \qquad (1 \leqslant \nu \leqslant \nu_{\max}(h_1)).$$

E.g., the first inequality follows from

$$\begin{aligned} |L_lG_l^\nu|_{0,-2m} &\leqslant |L_lD_l^{-1}D_l'|_{0,-2m}|D_l'^{-1}D_lG_l^\nu|_{0,0} \\ &\leqslant |D_l'^*D_l^{*-1}L_l^*|_{2m,0}\|D_l'^{-1}D_lG_l^\nu\| \leqslant C. \end{aligned}$$

(4) Let γ be as in Lemma 2. By (3.6b) $|L_l'^{-1}|_{\vartheta-m,\vartheta+m} \leqslant C$ holds for some $0 < \vartheta < 1/2$. Lemma 2 proves $|\gamma|_{\vartheta+m,0} \leqslant Ch_l^{\vartheta+m}$. Interpolation of (4.4a) and (4.4b*) yields $|L_l' G_l'^{\nu}|_{0,\vartheta-m} \leqslant Ch_l^{-m-\vartheta}/(\nu+1)^{\beta}$ with $\beta = (m+\vartheta)/(2m)$. Therefore,

$$\|\gamma G_l'^{\nu}\| \leqslant |\gamma|_{\vartheta+m,0}|L_l'^{-1}|_{\vartheta-m,\vartheta+m}|L_l'G_l'^{\nu}|_{0,\vartheta-m} \leqslant C/(\nu+1)^{\beta}$$

is valid. Applying again the Perron-Frobenius theory, we obtain

$$\text{(4.13e)} \qquad \|\gamma G_l^{\nu}\| \leqslant C/(\nu+1)^{\beta}, \quad \beta = (m+\vartheta)/(2m) \qquad (0 \leqslant \nu \leqslant \nu_{\max}(h_1))$$

from (4.13b). Now,

$$\text{(4.13f)} \qquad \|L_l'' G_l^{\nu}\| \leqslant Ch_l^{-2m}/(\nu+1)^{\beta}, \quad \beta = (m+\vartheta)/(2m) \qquad (1 \leqslant \nu \leqslant \nu_{\max}(h_1))$$

can be concluded from $\|L_l''G_l^{\nu}\| = \|L_l''G_l\|\,\|\gamma G_l^{\nu-1}\|$, since there is γ satisfying the conditions of Lemma 2 with $L_l''G_l = L_l''G_l\gamma$. The second term is estimated in (4.13e). Split the first term into $L_lG_l - L_l'G_l$. $\|L_lG_l\| \leqslant Ch_l^{-2m}|L_lG_l|_{0,-2m} \leqslant Ch_l^{-2m}$ follows from (4.13d) ($\nu = 1$). (4.4b*) for L_l' and (4.13b) yield $\|L_l'G_l\| \leqslant Ch_l^{-2m}$.

(5) Using $L_l'(G_l'^{\nu} - G_l^{\nu}) = -\sum_{\mu=0}^{\nu-1} L_l'G_l'^{\mu}G_l''G_l^{\nu-\mu-1}$ and $G_l'' = G_l''\gamma$, one obtains

$$\|L_l'(G_l'^{\nu} - G_l^{\nu})\| \leqslant \sum_{\mu=0}^{\nu-1} \|L_l'G_l'^{\mu}\|\,\|G_l''\|\,\|\gamma G_l^{\nu-\mu-1}\|$$

$$\leqslant h_l^{-2m}C' \sum_{\mu=0}^{\nu-1} [(\mu+1)^{-1}(\nu-\mu)^{-\beta}] \leqslant Ch_l^{-2m}/(\nu+1)^{\vartheta/m}.$$

This estimate, (4.4a) (for L_l'), and (4.13f) yield (4.4a) for L_l:

$$\|L_lG_l^{\nu}\| \leqslant \|L_l''G_l^{\nu}\| + \|L_l'(G_l^{\nu} - G_l'^{\nu})\| + \|L_l'G_l'^{\nu}\| \leqslant Ch_l^{-2m}/(\nu+1)^{\vartheta/m} \qquad (1 \leqslant \nu \leqslant \nu_{\max}(h_1)).$$

Repeating the proof of Criterion 3* yields (2.6). □

Example. Consider the Shortley-Weller discretization L_l (cf. last example of Section 3). L_l' is the usual five-point formula. Hence, (4.4a) and (4.4b*) are fulfilled with $\nu_{\max}(h) = \infty$. (3.6b) holds for all $\theta_0 = \theta_0' < 1/2$. Also the conditions (4.12a, b) are satisfied. Thus, the smoothing property holds for all ν ($\nu_{\max} = \infty$).

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