Inductive Formulae for General Sum Operations

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Abstract. In this note we report some computer generated formulae for the sum of powers of numbers with nonunitary increments; these reduce to the well-known cases when the increment is one.

1. Introduction. Inductive formulae for sums of powers of consecutive integers are well known; the left side of Table 1, based on [1], depicts such formulae for powers up to 10. These are usually derived directly from the fundamental theorem of sum calculus, or via the Bernoulli polynomials as

\[ \sum_{i=0}^{m-1} (1 + i)^n = \frac{1}{n+1} [B_{n+1}(m+1) - B_{n+1}] . \]

Formulae for the case where the increment is not one, cannot apparently be found explicitly in the literature; [4]–[10]. Conceptually these formulae are simple to obtain; however, the algebraic manipulations required tend to be overwhelming. In this note we present the first ten formulae, as obtained on a computer by formal string manipulations.

2. Approach. Let

\[ S^n(1, d, m) = \sum_{i=0}^{q} (1 + id)^n \]

where \( q = (m - 1)/d \) is an integer, \( d > 0 \). We desire a formal closed-form expression for \( S^n(1, d, m) \). Clearly

\[ S^n(1, d, m) = 1 + \sum_{k=0}^{n} \left( \binom{n}{k} d^{n-k} + \sum_{k=0}^{n} \left( \binom{n}{k} (2d)^{n-k} + \ldots + \sum_{k=0}^{n} \left( \binom{n}{k} (qd)^{n-k} \right) \right) \right) \]

\[ = \frac{m-1}{d} + \sum_{k=0}^{n-1} \left( \binom{n}{k} \left[ d^{n-k} + 2^{n-k} d^{n-k} + \ldots + q^{n-k} d^{n-k} \right] \right) \]

\[ = \frac{m-1}{d} + \sum_{k=0}^{n-1} \left( \binom{n}{k} d^{n-k} \sum_{j=1}^{q} j^{n-k} \right) \]

The individual terms in the second expression are indeed the entries of the left side of Table 1. The only remaining task is obtaining a formal expression for the first summation, by collecting appropriate terms; this is a rather long and tedious task, particularly for high values of \( n \).

Received December 11, 1978.


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0025-5718/80/0000-0062/$01.75

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**Table 1**

**Inductive formulae for general sum operations**

| $s^1(1,1,m)$ | $= \frac{m}{2}$ (m+1) |
| $s^2(1,1,m)$ | $= \frac{m}{6}$ (m+1)(2m+1) |
| $s^3(1,1,m)$ | $= \frac{m^2}{4}$ (m+1)$^2$ |
| $s^4(1,1,m)$ | $= \frac{m}{30}$ (m+1)(2m+1)(3m$^2$+3m-1) |
| $s^5(1,1,m)$ | $= \frac{m^2}{12}$ (m+1)$^2$ (2m$^2$+2m-1) |
| $s^6(1,1,m)$ | $= \frac{m}{42}$ (m+1)(2m+1)(3m$^4$+6m$^3$-3m+1) |
| $s^7(1,1,m)$ | $= \frac{m^2}{24}$ (m+1)$^2$ (3m$^4$+6m$^3$-m$^2$-4m+2) |
| $s^8(1,1,m)$ | $= \frac{m}{90}$ (m+1)(2m+1)(5m$^6$+15m$^5$+5m$^4$-15m$^3$+9m-3) |
| $s^9(1,1,m)$ | $= \frac{m^2}{20}$ (m+1)$^2$ (2m$^6$+6m$^5$+m$^4$-8m$^3$+2m$^2$+6m-3) |
| $s^{10}(1,1,m)$ | $= \frac{m}{66}$ (m+1)(2m+1)(3m$^8$+12m$^7$+8m$^6$-18m$^5$-10m$^4$+24m$^3$+2m$^2$-15m+5) |

| $s^1(1,d,m)$ | $= \frac{1}{48}$ (m+1+d) (m+1) |
| $s^2(1,d,m)$ | $= \frac{1}{60}$ [m(m+d)(2m+d) - (d-1)(d-2)] |
| $s^3(1,d,m)$ | $= \frac{1}{48}$ [(m+1+d)(m+1)(m$^2$+m+1-d)] |
| $s^4(1,d,m)$ | $= \frac{m^3-1}{60} + \frac{m^2+1}{2} + \frac{d}{30} (m^3-1) - \frac{d^3}{30}$ (m-1) |
| $s^5(1,d,m)$ | $= \frac{m^3-1}{60} + \frac{m^2+1}{2} + \frac{5d}{12} (m^4-1) - \frac{d^3}{12}$ (m-1) |
| $s^6(1,d,m)$ | $= \frac{m^3-1}{60} + \frac{m^2+1}{2} + \frac{d}{6} (m^5-1) - \frac{d^3}{6}$ (m$^3$-1) + \frac{d^5}{42}$ (m-1) |
| $s^7(1,d,m)$ | $= \frac{m^3-1}{60} + \frac{m^2+1}{2} + \frac{d}{6} (m^5-1) - \frac{d^3}{6}$ (m$^3$-1) + \frac{d^5}{42}$ (m-1) |
| $s^8(1,d,m)$ | $= \frac{m^3-1}{60} + \frac{m^2+1}{2} + \frac{d}{6} (m^5-1) - \frac{d^3}{6}$ (m$^3$-1) + \frac{d^5}{42}$ (m-1) |
| $s^9(1,d,m)$ | $= \frac{m^3-1}{60} + \frac{m^2+1}{2} + \frac{d}{6} (m^5-1) - \frac{d^3}{6}$ (m$^3$-1) + \frac{d^5}{42}$ (m-1) |
| $s^{10}(1,d,m)$ | $= \frac{m^{10}-1}{110} + \frac{m^9+1}{2} + \frac{5d}{6} (m^9-1) - \frac{d^3}{6} (m^7-1) + \frac{d^5}{10} (m^5-1) - \frac{d^7}{10}$ (m$^3$-1) + \frac{d^9}{66}$ (m-1) |
The algebraic manipulations have been carried out by a computer program. CPU time on a dedicated DEC PDP 11/70 was 2 hours; the code consisted of about 400 statements. The results are depicted on the right-hand side of Table 1. We now have closed-form expressions for summations such as \( \sum_j (1 + j\sqrt{A})^2 \) or \( \sum_j (1 + j\pi)^3 \).

3. Related Facts.

Fact 1. Besides brute force computation, the results of Table 1 may be proved by induction on \( q \), for a fixed \( d \), and \( n \).

The following facts can also be proved.

Fact 2. For all \( n, d \),

\[
S^n(1, d, m) \approx \frac{m^{n+1} - 1}{(n + 1)d} + \frac{m^n + 1}{2} + \frac{nd}{12} (m^{n-1} - 1),
\]

which is exact for \( n = 1 \).

Fact 3. For sufficiently small \( d \), \( f(x) \) Riemann integrable, \( m \equiv 1 \mod d \), and

\[
\Lambda f(1, d, m) = \sum_{i=0}^{(m-1)/d} f(1 + id),
\]

there exists a \( \delta \) such that

\[
\left| \Lambda f(1, d, m) - \frac{1}{d} \int_1^m f(x) \, dx \right| < \delta.
\]

In particular, if \( f(x) = x^n \)

\[
\left| S^n(1, d, m) - \frac{m^{n+1} - 1}{(n + 1)d} \right| < \delta.
\]

This is related to the Euler-Maclaurin sum formula [2], and a result on the generalized factorial. [3].

Fact 4. The sum of the odd integers up to \( m \) is equal to the sum of the cubes of all integers up to \( m \), divided by \( m^2 \); namely, \( S^3(1, 1, m)/m^2 = S^1(1, 2, m) \).

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