Exponential Laws for Fractional Differences

By Godfrey L. Isaacs

Abstract. In Math. Comp., v. 28, 1974, pp. 185—202, Diaz and Osler gave the following (formal) definition for \( \Delta^\alpha f(z) \), the \( \alpha \)th fractional difference of \( f(z) \): 

\[
\Delta^\alpha f(z) = \sum_{p=0}^{\infty} A_{p}^{-\alpha-1} f(z + \alpha - p),
\]

where \( A_{p}^{-\alpha-1} = (p^{-\alpha-1}) = (-1)^p (\alpha)^p \). (Note: in [2] \( \Delta^\alpha \) is written \( \Delta^\alpha \).)

Since \( A_{p}^{-\alpha-1} = O(p^{-\alpha-1}) \) as \( p \to \infty \), the series is convergent for every \( z \), if \( f(t) = O(t^{-e}) \) as \( |t| \to \infty \). Diaz and Osler show [2, p. 189], that if \( z \) and \( \alpha \) are fixed and if (in addition to the order condition above) \( f(t) \) is analytic in a region \( R \) containing the points \( t = z + \alpha - p, p \geq 0 \), then \( \Delta^\alpha f(z) \) may be put in the form of a line integral round a contour in \( R \). They ask [2, p. 201] whether there is an exponent law for \( \Delta^\alpha f(z) \) of the form

\[
\Delta^\alpha f(z) = \Delta^\alpha \Delta^\beta f(z).
\]

If \( s_n = f(n) \), we obtain formally, for the sequence \( s_n \),

\[
\Delta^\alpha s_n = \sum_{p=0}^{\infty} A_{p}^{-\alpha-1} s_{n+p},
\]

If \( \alpha = 0, 1, 2, \ldots \), the series terminates at \( p = \alpha \), and gives successive "backward differences," starting (at \( \alpha = 1 \)) with the difference \( \Delta^1 s_n = s_{n+1} - s_n \).

2. An Exponent Law. In [3] the following definition for the \( \alpha \)th fractional difference of a sequence \( s_n \) was used:

\[
\Delta^\alpha s_n = \sum_{p=0}^{\infty} A_{p}^{-\alpha-1} s_{n+p},
\]

the series being supposed summable in some Cesàro sense. The definition is due to

Received January 2, 1979.

1980 Mathematics Subject Classification. Primary 33A70, 40A05; Secondary 65N10;
Key words and phrases. Fractional differences, successive differences, exponent law, summability of series.
Chapman [1]. For \( \alpha = 0, 1, 2, \ldots \), the series terminates at \( p = \alpha \) and we get successive “forward differences” starting (at \( \alpha = 1 \)) with the difference \( \Delta^1 s_n = s_n - s_{n+1} \). In fact, as is easily verified,

\[
\Delta^\alpha s_n = (-1)^\alpha \hat{\Delta}^\alpha s_n \quad (\alpha = 0, 1, 2, \ldots).
\]

If \( \alpha \) is fractional, the formula (3) fails to make sense, since \( \hat{\Delta}^\alpha s_n \) takes \( s_n \) off its domain; further, (5) is no help since \((-1)^\alpha\) is neither real nor unique.

In [3, Theorem 1] the following exponent formula was obtained for the fractional differences (4):

\[
\Delta^{\lambda_1 s_n} = \Delta^{\lambda_1 s_n} \tag{6}
\]

where \( \lambda > -1, \lambda + s > -1, r + s \neq 0, 1, 2, \ldots, e = 0 \) or \( > 0 \) according to whether \( s \) is or is not an integer, and (unfortunately) \( r < 0 \) in the case \( s \neq 0, 1, 2, \ldots \). Here it is assumed that the left side is summable \((C, \lambda)\). (The series giving \( \Delta^s s_n \) is then automatically summable \((C, \mu)\), where \( \mu \geq \max(\lambda + r, -1) \).)

Because of the failure to relate the definitions \( \Delta^\alpha s_n \) and \( \hat{\Delta}^\alpha s_n \) in the case \( \alpha \neq 0, 1, 2, \ldots \), it did not seem likely that (6) could be of help in finding an exponent law of the type (2). However, if we write (2) out formally we obtain

\[
\sum_{p=0}^{\infty} A_p r^{-p-1} f(z + r + s - p) = \sum_{k=0}^{\infty} A_k r^{-k-1} \sum_{m=0}^{\infty} A_m s^{-m} f(z + r + s - k - m), \tag{7}
\]

and if we write (6) out, we get

\[
\sum_{p=0}^{\infty} A_p r^{-p-1} s_{n+p} = \sum_{k=0}^{\infty} A_k r^{-k-1} \sum_{m=0}^{\infty} A_m s^{-m} s_{n+k+m} \tag{8}.
\]

We see that in (7) the same values of \( f \) are used on both sides, namely \( f(z + r + s - q) \), where \( q = 0, 1, 2, \ldots \), the jump from \( f(z) \) to \( f(z + s - m) \) occasioned by \( \hat{\Delta}^s \) being overlaid by the subsequent jump due to \( \hat{\Delta}^r \). Thus, if we put

\[
s_q = f(z + r + s - q) \quad (q = 0, 1, 2, \ldots) \tag{9}
\]

in (8), with \( n = 0 \), we obtain (7). We have thus obtained the following exponent law for Diaz and Osier’s differences:

**Theorem 1.** Let \( \lambda > -1, \lambda + s > -1, r + s \neq 0, 1, 2, \ldots, e = 0 \) or \( > 0 \) according as \( s \) is integral or fractional, and \( r < 0 \) if \( s \neq 0, 1, 2, \ldots \). Then

\[
\hat{\Delta}^{\lambda_1 s_n} f(z) = \hat{\Delta}^{\lambda_1 s_n} \hat{\Delta}^s f(z), \tag{10}
\]

under the assumption that the left side is summable \((C, \lambda)\).

3. A “Converse” Exponent Law. In [3, Theorem 3] a “converse” result to (6) is given, which in its “convergence” form [3, Theorem 3'] is as follows:

\[
\Delta^{\lambda_1 s_n} s_n = \Delta^{\lambda_1 s_n} \Delta^s s_n \tag{11}
\]

the two right side series being assumed convergent. Here (apart from the trivial cases \( r = 0 \) or \( s = 0 \)) \( r \) and \( s \) must be in the first or fourth quadrant or inside the open triangles with vertices \((0, k), (0, k + 1), (-1, k + 1), k = 0, 1, 2, \ldots \). From this we obtain the corresponding formula for Diaz and Osier’s differences:
Theorem 2. If \( r \) and \( s \) are in the set \( S \) just described,

\[
\hat{\Delta}_{(c,0)}^{r+s}f(z) = \hat{\Delta}_{(c,0)}^{r}A_{(c,0)}^{s}f(z),
\]

the two right side series being supposed convergent.

The last formula is useful in extending known results of Diaz and Osier. In [2, Table 2.1], they give \( \hat{\Delta}^{\alpha}f(z) \) for some special functions \( f(z) \). In each case it can be seen that the two series on the right side of (12) are convergent for the value of \( \alpha (= s) \) given, and for \( r \) and \( s \) in the set \( S \); hence, we know that the \( r \)th difference of the expression \( \hat{\Delta}^{\alpha}f(z) \) \((\alpha = s)\) given in the table is just the difference \( \hat{\Delta}^{r+s} \) of the function \( f(z) \). In short, the functions \( f(z) \) given in the table all satisfy the exponent law (12) with suitable restrictions on \( r \) and \( s \).

4. An Example. As an example of the above, let

\[
f(z) = z^{(p)} = \frac{\Gamma(z + 1)}{\Gamma(z + 1 - p)}.
\]

Then by [2, Table 2.1], with \( s \) for \( \alpha \),

\[
\hat{\Delta}^{s}f(z) = \frac{\sin(\pi z)\Gamma(s - p)z^{(p-s)}}{\sin(\pi(z + s))\Gamma(-p)}
\]

for \( s > p \). (It is assumed that both \( z^{(p)} \) and \( \hat{\Delta}^{s}f(z) \) are defined by continuity at points of removable singularity, and that \( z, p, s \) are chosen so as to avoid points of unremovable singularity in either of them; thus if, in \( z^{(p)} \), \( z \) is a negative integer, \( z - p \) must be, and if, in \( \hat{\Delta}^{s}f(z) \), \( z + s \) is an integer, then \( p = 0 \) or a positive integer.) Now

\[
\hat{\Delta}_{k}^{r+s}f(z) = \sum_{k=0}^{\infty} A_{k}^{r+s}f(z + r - k).
\]

Replacing \( z \) by \( z + r - k \) in (13), we see that \( \hat{\Delta}^{s}f(z + r - k) \) is \( O(|z| + k)^{p-s} \) as \( k \to \infty \). Hence, since \( A_{k}^{r+s} = O(k^{r+s}) \), the series in (14) converges if \( r + s > p \).

(To avoid unremovable singularities in the terms of the series of (14) we see that if, for any \( k \), \( z + r - k + s \) is an integer, then we must take \( p = 0 \) or a positive integer; and it is gratifying to see that this happens if and only if, whenever \( z + r + s \) is an integer, then \( p = 0, 1, 2, \ldots \), which is the criterion that \( \hat{\Delta}^{r+s}f(z) \) has no unremovable singularity.)

Hence the equality in (12) is true for \( f(z) = z^{(p)} \) with \( s > p, r + s > p \), and \( r, s \) in the set \( S \) (and, of course, \( p = 0, 1, 2, \ldots \), if \( z + r + s \) happens to be an integer). In particular, if \( p > 0 \) and \( z + r + s \) is nonintegral, (12) is true if \( s > p \) and \( r > 0 \), a useful case. The arguments for the other functions \( f(z) \) of Table 2.1 are similar.

5. Relation Between \( \Delta^{\alpha} \) and \( \hat{\Delta}^{\alpha} \). Although there is no extension of the Diaz-Osler differences (1) to sequences, for \( \alpha \) fractional, there is an immediate extension of the differences (4) to functions \( f(z) \):

\[
\Delta^{\alpha}f(z) = \sum_{\rho=0}^{\infty} A_{\rho}^{-\alpha-1}f(z + \rho).
\]

Diaz and Osler ask [2, p. 201] whether there is a relation between (1) and other dif-
ferences. Now for \( \alpha = 0, 1, 2, \ldots \), we can replace \( \infty \) in (1) by \( \alpha \) and then replace \( p \) by \( \alpha - p \). This shows that by (15),

\[
\Delta^\alpha f(z) = (-1)^\alpha \Delta f(z).
\]

But as with \( s_n \) in (5), this has no meaning for \( \alpha \) fractional.

Let us write for given fixed \( z \) and \( \alpha \),

\[
g(u) = f(2z + \alpha - u).
\]

Then it is easy to verify:

**Theorem 3.** If the series for either side converges, then

\[
\Delta^\alpha f(z) = (\Delta^\alpha g(u))_{u=z}
\]

where \( g(u) \) is given by (17).

This enables us to calculate \( \Delta^\alpha f(z) \) from known differences of the \( \Delta \) type. For example, let

\[
f(z) = z(\rho) = \frac{\Gamma(z + 1)}{\Gamma(z - p + 1)}.
\]

Then

\[
g(u) = f(2z + \alpha - u) = (2z + \alpha - u)(\rho) = \frac{\Gamma(2z + \alpha + 1 - u)}{\Gamma(2z + \alpha + 1 - p - u)} = \frac{\Gamma(A - u)}{\Gamma(B - u)},
\]

say. Thus by [2, Table 2.1, #4],

\[
\Delta^\alpha f(z) = (\Delta^\alpha g(u))_{u=z} = \left( \frac{\Gamma(B - A + \alpha)\Gamma(A - \alpha - u)}{\Gamma(B - A)\Gamma(B - u)} \right)_{u=z} (B - A > -\alpha)
\]

\[
= \frac{\Gamma(-p + \alpha)\Gamma(2z + 1 - u)}{\Gamma(-p)\Gamma(2z + \alpha + 1 - p - u)}_{u=z}
\]

\[
= \frac{\Gamma(\alpha - p)\Gamma(z + 1)}{\Gamma(-p)\Gamma(z + \alpha + 1 - \rho)}
\]

\[
= \frac{\Gamma(\alpha - p)}{\Gamma(-p)} z^{(p-a)}
\]

for \( \alpha > p \).

By (13) this gives

\[
\Delta^\alpha f(z) = \frac{\sin(\pi(z + \alpha))}{\sin(\pi z)} \Delta^\alpha f(z)
\]

when \( \alpha > p \), which is a direct extension of (16) to fractional values of \( \alpha \).